Report of the Forty First
Canadian Mathematical Olympiad
2009

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The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Canadian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Sunlife Financial Canadian Open Mathematics Challenge (COMC). This year, the top 50 COMC scores were invited outright to write the CMO. Approximately 150 others, next in rank, were invited to send solutions to a Repêchage set of ten problems posted on-line within a week to the University of Waterloo. Twenty-five students were invited on this basis. I am grateful to Ian VanderBurgh for setting this up and assembling a team of markers, consisting of Fiona Dunbar, Mike Eden, Matthew Faubert, Steve Furino, Jeremy Kuikman, Jen Nissen, J.P. Pretti, Jim Schurter, Ian VanderBurgh and Troy Vasiga, to go through the 95 scripts received.

The Society is grateful for support from **Sun Life Financial** and the other sponsors listed on the previous page**.**

I am very grateful to the CMO Committee members who submitted problems, reviewed the test and marked the solutions: Andrew Adler, Edward Barbeau, Jason Bell, Julia Gordon, Robert Morewood, Zinovy Reichstein, Naoki Sato and Jozsef Solymosi. Thanks also go to Richard Hoshino for suggesting problems for the test and to Joseph Khoury for translating the paper into French. I am indebted to the previous CMO Chair Edward Barbeau for guiding me through the process, to Solange Hupé for the hard work done at CMS headquarters and to the Executive Director, Graham Wright, for making sure we were all on track.

Kalle Karu, Chair
Canadian Mathematical Olympiad Committee

The 41st (2009) Canadian Mathematical Olympiad was written on Wednesday, March 25, 2009. A total of 82 students were invited and 81 sets of solutions were received. Of these, 68 students were deemed eligible for official CMO participation. Six Canadian provinces were represented, with the number of contestants as follows:

BC (10)   AB (8)   SK (1)   MB (2)   ON (41)   QC (3)

The 2009 CMO consisted of five questions, each marked out of 7. The maximum score attained by a student was 30. The official contestants were grouped into four divisions according to their scores as follows:

Division	Range of Scores	No. of Students
I	25 - 30	8
II	20 - 24	10
III	11 - 18	20
IV	0 - 10	30

The following tables give the scores obtained on the COMC along with the corresponding scores obtained by candidates on the CMO. Students obtaining scores 73-80 qualified directly to write the CMO; students with scores 67-72 qualified through the Repêchage. The group "Other" includes students who wrote the Repêchage but not the COMC, or were invited based on other high-school competition results.

80 (28, 21)
79 (29, 25)
78 (30, 28, 24, 11)
77 (15)
76 (23, 22, 21, 18, 15, 12, 2, 0)
75 (28, 20, 18, 17, 8, 2)
74 (24, 14, 8, 6, 0)
73 (16, 16, 12, 8, 2, 1, 0, 0)

72 (16)
71 (9)
70 (23, 22)
69 (17, 14, 12, 5)
68 (13, 9, 8, 7, 4, 3)
67 (16, 12, 8, 7, 3, 2, 0)

Other (25, 25, 22, 17, 13, 10, 8, 5, 4, 1, 0)

FIRST PRIZE — Sun Life Financial Cup — $2000

Jonathan Schneider
University of Toronto Schools, Toronto, Ontario

SECOND PRIZE — $1500

XiaoLin (Danny) Shi
Sir Winston Churchill High School, Calgary, Alberta

THIRD PRIZE — $1000

Chengyue (Jarno) Sun
Western Canada High School, Calgary, Alberta

HONOURABLE MENTIONS — $500

Ahmad Abdi
Sabouhi Academy of Art & Design
Toronto, Ontario

Yu (Robin) Cheng
Pinetree Secondary School
Coquitlam, British Columbia

Nikita Lvov
Marianopolis College
Westmount, Quebec

Hunter Spink
Western Canada High School
Calgary, Alberta

Chen Sun
A.B. Lucas Secondary School
London, Ontario

Division 1
25-30

Jonathan Schneider	University of Toronto Schools	ON
Danny Shi	Sir Winston Churchill H.S.	AB
Jarno Sun	Western Canada H.S.	AB
Ahmad Abdi	Sabouhi Academy of Art & Design	ON
Robin Cheng	Pinetree S.S.	BC
Nikita Lvov	Marianapolis College	QC
Hunter Spink	Western Canada H.S.	AB
Chen Sun	A.B. Lucas S.S.	ON

Division 2
20-24

Joshua Alman	University of Toronto Schools	ON
Fang Guo	Richmond Hill H.S.	ON
Neil Gurram	ICAE	MI
Alan Huang*	ICAE	MI
Heinrich Jiang	Vincent Massey S.S.	ON
Zhiqiang Liu	Don Mills C.I.	ON
Mariya Sardarli	McKernan J.H.S.	AB
Alex Song	Waterloo C.I.	ON
Weinan Peter Wen	Vincent Massey S.S.	ON
Jonathan Zhou	Pinetree S.S.	BC
Jonathan Zung	University of Toronto Schools	ON

Division 3
11-18

Mohammad Babadi*	Thornlea S.S.	ON
Frank Ban	Vincent Massey S.S.	ON
Ram Bhaskar*	ICAE	MI
Ruiyuan Chen	Gleneagle S.S.	BC
Yuhan Chen	Sir Winston Churchill C.V.I	ON
Yuzhou Chen	Sir John A MacDonald C.I.	ON
Earl Chua*	Math. Train. Guild of the Philippines	
Andrew Dhawan	The Woodlands School	ON
Rong Fu	A.Y. Jackson S.S.	ON
Leo Guo	Dr. Norman Bethune C.I.	ON
Zhebin Hu	Marianopolis College	QC
Junghoo Kim	University Hill S.S.	BC
Victor Liu	Dr. Norman Bethune C.I.	ON
Sudharshan Mohanram*	ICAE	MI
Serguei Makarov	The Abelard School	ON
Ryan Peng	Centennial Collegiate	SK
James Rickards	Colonel By S.S.	ON
Julian Sun	Sir Winston Churchill S.S.	BC
John Russell Virata*	Math. Train. Guild of the Philippines	
Jixuan Wang	Don Mills C.I.	ON
Richard Wang	Sir Winston Churchill S.S.	BC
Ming Jing Wong	A.B. Lucas S.S.	ON
Daniel Yoo	Thornhill S.S.	ON

Division 4
0-10

Golam Tahrif Bappi	Waterloo C.I.	ON
Brian Bi	Woburn C.I.	ON
Cheuk Ho Choi	St. George's School	BC
Naiwen Cui	Waterloo C.I.	ON
Yuri Delanghe	Harry Ainlay Senior H.S.	AB
Kun Dong*	Sir William Mulock S.S.	ON
James Duyck	Vincent Massey S.S.	ON
Henry Fung*	Glenforest S.S.	ON
Jun Hou Fung	Canadian Int'l School of Hong Kong	HK
Daniel Galperin	Waterloo C.I.	ON
Chengcheng Gui	St. John's-Ravenscourt School	MB
Adam Halski*	Kuwait English School	KW
James Hayeur	Vanier College	QC
Robin He*	ICAE	MI
Siyang He	Waterloo C.I.	ON
Jingjie Hu*	London Int'l Academy	ON
Amlesh Jakakumar	Waterloo C.I.	ON
Kaifan Liuzhao	David and Mary Thomson C.I.	ON
Anupa Murali	Bishop Brady H.S.	NH
Xiaoqi Shi	Agincourt C.I.	ON
Xing Shuo	Western Canada H.S.	AB
Suraj Srinivasan	Fort Richmond C.I.	MB
Paul Sticea*	Vanier College	QC
Tanya Tang	Sir Winston Churchill S.S.	BC
Junho Whang	East Northumberland S.S.	ON
Zhongyi Wan	Agincourt C.I.	ON
Brent Wang	Emily Carr	ON
Hanson Wang	Woburn C.I.	ON
Susan Wang*	Burnaby Central S.S.	BC
Sherwin Wu	ICAE	MI
Chuan Xin (Oliver)	Matthew McNair S.S.	BC
Carrie Xing	Marc Garneau C.I.	ON
Xinlei Xu	Lord Byng S.S.	BC
Aaron Yang	Northern S.S.	ON
Xiaoye (Simon) Yin	O'Neill C.V.I.	ON
Anqi Zhang	Vincent Massey S.S.	ON
Zhen Zhang	Western Canada H.S.	AB
Pei Jun Zhao	London Central S.S.	ON
Kaiven Zhou	Old Scona Academic H.S.	AB

The students indicatd by * wrote the 2009 CMO as unofficial candidates.

PROBLEMS

PROBLEM 1. Given an $m \times n$ grid with squares coloured either black or white, we say that a black square in the grid is *stranded* if there is some square to its left in the same row that is white and there is some square above it in the same column that is white (see Figure 1.).
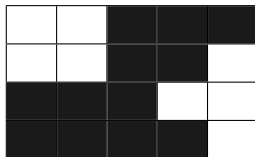

FIGURE 1. A 4×5 grid with no stranded black squares

Find a closed formula for the number of $2 \times n$ grids with no stranded black squares.

PROBLEM 2. Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

PROBLEM 3. Define
$$f(x, y, z) = \frac{(xy + yz + zx)(x + y + z)}{(x + y)(x + z)(y + z)}.$$
Determine the set of real numbers r for which there exists a triplet (x, y, z) of positive real numbers satisfying $f(x, y, z) = r$.

PROBLEM 4. Find all ordered pairs (a, b) such that a and b are integers and $3^a + 7^b$ is a perfect square.

PROBLEM 5. A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

CANADIAN MATHEMATICAL OLYMPIAD 2009
SOLUTIONS

PROBLEM 1. Given an $m \times n$ grid with squares coloured either black or white, we say that a black square in the grid is *stranded* if there is some square to its left in the same row that is white and there is some square above it in the same column that is white (see Figure).

FIGURE 1. A 4×5 grid with no stranded black squares

Find a closed formula for the number of $2 \times n$ grids with no stranded black squares.

Solution. There is no condition for squares in the first row. A square in the second row can be black only if the square above it is black or all squares to the left of it are black. Suppose the first k squares in the second row are black and the $(k+1)$-st square is white or $k = n$. When $k < n$ then for each of the first $k + 1$ squares in the first row we have 2 choices, and for each of the remaining $n - k - 1$ columns we have 3 choices. When $k = n$, there are 2^n choices for the first row. The total number of choices is thus:

$$\sum_{k=0}^{n-1} 2^{k+1} 3^{n-k-1} + 2^n.$$

This expression simplifies to

$$2 \cdot 3^n - 2^n.$$

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

PROBLEM 2. Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

Solution. Let $x_0, \ldots, x_{199}$ be variables. Assign the value of $+1$ or -1 to x_i depending on whether the $(i+1)$st segment of the larger circle (counting counterclockwise) is black or white, respectively. Similarly, assign the value of $+1$ or -1 to the variable y_i depending on whether the $(i+1)$th segment of the smaller circle is black or white. We can now restate the problem in the following equivalent way: show that

$$S_j = \sum_{i=1}^{200} x_i y_{i+j} \geq 0 \,,$$

for some $j = 0, \ldots, 199$. Here the subscript $i+j$ is understood modulo 200.
Now observe that $y_0 + \cdots + y_{199} = 0$ and thus

$$S_0 + \cdots + S_{199} = \sum_{I=0}^{199} x_i (y_0 + \cdots + y_{199}) = 0 \,.$$

Thus $S_j \geq 0$ for some $j = 0, \ldots, 199$, as claimed.

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

PROBLEM 3. Define
$$f(x, y, z) = \frac{(xy + yz + zx)(x + y + z)}{(x + y)(x + z)(y + z)}.$$
Determine the set of real numbers r for which there exists a triplet (x, y, z) of positive real numbers satisfying $f(x, y, z) = r$.

Solution. We prove that $1 < f(x, y, z) \leq \frac{9}{8}$, and that $f(x, y, z)$ can take on any value within the range $(1, \frac{9}{8}]$.
The expression for $f(x, y, z)$ can be simplified to
$$f(x, y, z) = 1 + \frac{xyz}{(x + y)(x + z)(y + z)}.$$
Since x, y, z are positive, we get $1 < f(x, y, z)$.
The inequality $f(x, y, z) \leq \frac{9}{8}$ can be simplified to
$$x^2 y + x^2 z + y^2 x + y^2 z + z^2 x + z^2 y - 6xyz \geq 0.$$
Rearrange the left hand side as follows:
$$x^2 y + x^2 z + y^2 x + y^2 z + z^2 x + z^2 y - 6xyz =$$
$$x(y^2 + z^2) - 2xyz + y(x^2 + z^2) - 2xyz + z(x^2 + y^2) - 2xyz =$$
$$x(y - z)^2 + y(x - z)^2 + z(x - y)^2.$$
This expression is clearly non-negative when x, y, z are non-negative.
To prove that $f(x, y, z)$ takes any values in the interval $(1, \frac{9}{8}]$, define
$$g(t) = f(t, 1, 1) = 1 + \frac{t}{2(1 + t)^2}.$$

Then $g(1) = \frac{9}{8}$ and $g(t)$ approaches 1 as t approaches 0. It follows from the continuity of $g(t)$ for $0 < t \leq 1$ that it takes all values in the interval $(1, \frac{9}{8}]$. (Alternatively, one can check that the quadratic equation $g(t) = r$ has a solution t for any number r in the interval $(1, \frac{9}{8}]$.)

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

PROBLEM 4. Find all ordered pairs (a, b) such that a and b are integers and $3^a + 7^b$ is a perfect square.

Solution. It is obvious that a and b must be non-negative.
Suppose that $3^a + 7^b = n^2$. We can assume that n is positive. We first work modulo 4. Since $3^a + 7^b = n^2$, it follows that

$$n^2 \equiv (-1)^a + (-1)^b \pmod 4.$$

Since no square can be congruent to 2 modulo 4, it follows that we have either (i) a is odd and b is even or (ii) a is even and b is odd.

Case (i): Let $b = 2c$. Then

$$3^a = (n - 7^c)(n + 7^c).$$

It cannot be the case that 3 divides both $n - 7^c$ and $n + 7^c$. But each of these is a power of 3. It follows that $n - 7^c = 1$, and therefore

$$3^a = 2 \cdot 7^c + 1.$$

If $c = 0$, then $a = 1$, and we obtain the solution $a = 1$, $b = 0$. So suppose that $c \geq 1$. Then $3^a \equiv 1 \pmod 7$. This is impossible, since the smallest positive value of a such that $3^a \equiv 1 \pmod 7$ is given by $a = 6$, and therefore all a such that $3^a \equiv 1 \pmod 7$ are even, contradicting the fact that a is odd.

Case (ii): Let $a = 2c$. Then

$$7^b = (n - 3^c)(n + 3^c).$$

Thus each of $n - 3^c$ and $n + 3^c$ is a power of 7. Since 7 cannot divide both of these, it follows that $n - 3^c = 1$, and therefore

$$7^b = 2 \cdot 3^c + 1.$$

Look first at the case $c = 1$. Then $b = 1$, and we obtain the solution $a = 2$, $b = 1$. So from now on we may assume that $c > 1$. Then $7^b \equiv 1 \pmod 9$. The smallest positive integer b such that $7^b \equiv 1 \pmod 9$ is given by $b = 3$. It follows that b must be a multiple of 3. Let $b = 3d$. Note that d is odd, so in particular $d \geq 1$.
Let $y = 7^d$. Then $y^3 - 1 = 2 \cdot 3^c$, and therefore

$$2 \cdot 3^c = (y - 1)(y^2 + y + 1).$$

It follows that $y - 1 = 2 \cdot 3^u$ for some positive u, and that $y^2 + y + 1 = 3^v$ for some $v \geq 2$. But since

$$3y = (y^2 + y + 1) - (y - 1)^2,$$

it follows that $3 \mid y$, which is impossible since $3 \mid (y - 1)$.

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

PROBLEM 5. A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

Solution. (For a finite set of points only.) Let D be a disk of smallest radius that covers all marked points. Consider the marked points on the boundary C of this disk. Note that if all marked points on C lie on an arc smaller than the half circle (ASTTHC for short), then the disk can be moved a little towards these points on the boundary and its radius can be decreased. Since we assumed that our disk has minimal radius, the marked points on its boundary do not lie on an ASTTHC.

If the two endpoints of a diagonal of D are marked, then D is the smallest disk containing these two points, hence must have radius at most 1.

If there are 3 marked points on C that do not lie on an ASTTHC, then D is the smallest disk covering these 3 points and hence must have radius at most 1. (In this case the triangle formed by the three points is acute and C is its circumcircle.)

If there are more than 3 marked points on the boundary that do not lie on an ASTTHC, then we can remove one of them so that the remaining points again do not lie on an ASTTHC. By induction this leads us to the case of 3 points. Indeed, given 4 or more points on C, choose 3 points that lie on a half circle. Then the middle point can be removed.

THE GRADERS' REPORT

The 2009 CMO papers were graded by Andrew Adler, Jason Bell, Julia Gordon, Kalle Karu, Robert Morewood, Zinovy Reichstein and Jozsef Solymosi. All 81 papers were graded twice and the papers that received at least 20 marks were then reviewed once more.

The difficulty of the problems in this year's test was as follows. The first three problems were relatively easy, with approximately 50% students able to solve them. The fourth problem was harder, with few correct solutions but many partial marks. The last problem was very hard, with less than 10% of students getting close to a solution. Even though the test in general was slightly easier than last year, the mix of easier and harder problems did a fairly good job of separating out the participants.

For future tests the goal should be to include two easier problems similar to problems 1-3, then two harder problems and one very difficult problem. In this year's test problem 3 was intended to be one of the harder problems, but it turned out to be easier than expected.

Another feature of this year's test was the absence of any problems requiring classical Euclidean geometry. Students expect to see Euclidean geometry problems and are in general well-prepared for these. Many were trying to use Euclidean and analytic geometry tools to solve the last problem in the test. The first four problems rely on standard techniques, such as the pigeonhole principle, AM-GM inequality, modular arithmetic. These topics are all covered in high-school curriculum and it is only a question of applying them correctly. Students who have prepared for the CMO certainly have encountered many similar problems before.

Each question was marked out of 7 marks. The maximum score obtained was 30. The distribution of marks for each problem is given in the table:

	0	1	2	3	4	5	6	7
#1	34	0	3	5	5	6	3	25
#2	39	4	1	0	0	0	5	32
#3	23	5	10	8	2	2	15	16
#4	46	7	10	4	7	1	1	5
#5	59	11	4	0	1	2	3	1

Problem 1. The problem of counting configurations of squares looks very similar to the counting problems in past tests. However, this year's problem was easier because it did not require solving a linear recursion. One can of course set up a two-variable linear recursion relation, but solving it turns out to be very simple because the two variables uncouple. This was the approach used by most students, probably because of having studied previous CMO tests. Some students were confused with the terminology "to the left" and assumed it to mean "immediately to the left". This variation of the problem is much harder and actually does require solving a linear recurrence. Full marks were given for a correct solution of this version of the problem.

Problem 2. Many students were familiar with this type of problem. A typical solution was to count the total number of matches over all 200 possible rotations and then use the pigeonhole principle. Equivalently, one can calculate the average number of matches for a rotation to be 100. Both of these approaches as well as the algebraic approach given above appeared in the solutions.

Problem 3. The inequality problem was intended as a harder problem because students first had to find the bounds and then prove that these bounds are correct. Reading the solutions, it appears that most students are very skillful in applying the AM-GM inequality in its various forms and did not have much difficulty in proving the upper bound inequality. The Muirhead inequality was mentioned in a few papers. The second half of the problem (showing that every value in the interval is achieved) can be solved by either using the continuity of the function or by showing that a quadratic equation has a solution. As can be expected, few students knew about continuous functions, but almost everybody who understood the problem could solve the quadratic equation.

Problem 4. The number theory problem was the longest one and also the hardest one to grade. It was also very suitable for assigning partial marks. Each step of the proof is actually very simple, only involving divisibility and modular arithmetic. To get full marks, the students had to be persistent and go through all possible cases. It can be seen rather quickly that the proof divides into two main cases -- the easier case of b even and the slightly harder case of a even. The two cases were recognized by many students, but very few finished going through both cases.

Problem 5. The covering problem was the hardest one in the test. Only four students solved it or were close to solving it. There were, however, many attempts at solution. It was rather difficult to assign partial marks for these attempts that often lead to nowhere. The solution given above starts with a disk that covers all marked points. This was not the typical approach by students. Most students either tried to use induction on the number of marked points or tried to find three special points such that the minimal disk that covers these three points also covers the other points. All these approaches lead to a solution (and all 3 were represented among correct solutions), but the last two are longer and more involved. A typical error was forgetting that if a triangle is obtuse then its circumcircle is not the smallest circle bounding the triangle.