Report of the Thirty Fourth

Canadian Mathematical Olympiad

2002

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Canadian Mathematical Society

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Further information regarding the

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and the

Canadian Open Mathematics Challenge

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REPORT AND RESULTS OF THE 2002 CANADIAN MATHEMATICAL OLYMPIAD

The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Canadian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Olympiads Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Canadian Open Mathematical Challenge (COMC). Students may also be nominated to write the CMO by the provincial coordinators.

The Society is grateful for support from the Sun Life Financial as the Major Sponsor of the 2002 Canadian Mathematical Olympiad and the other sponsors which include: the Ministry of Education of Ontario, the Ministry of Education, Quebec, Alberta Learning, the Department of Education of New Brunswick, the Department of Education of Newfoundland and Labrador, the Department of Education of the Northwest Territories and the Department of Education of Saskatchewan; the Department of Mathematics and Statistics, University of New Brunswick at Fredericton, the Centre for Education in Mathematics and Computing, University of Waterloo, the Department of Mathematics and Statistics, University of Ottawa, the Department of Mathematics, University of Toronto, Nelson Thompson Learning and John Wiley and Sons Canada Ltd..

The provincial coordinators of the CMO are John Denton from Dawson College QC, Diane Dowling from University of Manitoba, Peter Crippin from University of Waterloo ON, Harvey Gerber from Simon Fraser University BC, Gareth J. Griffith from University of Saskatchewan, Ted Lewis from University of Alberta, Gordon MacDonald from University of Prince Edward Island, Roman Mureika from University of New Brunswick, Michael Nutt from Acadia University NS, Thrse Ouellet from Université de Montréal QC, and Donald Rideout from Memorial University of Newfoundland.

The CMO Subcommittee members and other mathematicians who set and/or marked the 2002 CMO papers were Robert Dawson, Saint Mary's University; Karl Dilcher, Dalhousie University; J. P. Grossman, Masschusets Institute of Technology; Richard Hoshino, Dalhousie University; Richard Lockhart, Simon Fraser University. Richard Nowakowski, Dalhousie University; Michael Nutt, Acadia University; Dorette Pronk, Dalhousie University; Naoki Sato, Sun Life, Toronto; Tony Thompson, Dalhousie University; Daryl Tingley, University of New Brunswick; Maureen Tingley, University of New Brunswick.

I would like to thank Professor Eric Marchand of the University of New Brunswick for the French translation of the 2002 CMO Examination and Solutions. As well as Caroline Baskerville of the CMS Executive Office and Linda Guthrie of the Mathematics and Statistics Department, University of New Brunswick, for a substantial amount of administrative work. Finally I must thank Graham Wright, the Executive Director of the CMS. It is because of Graham's devotion to the CMS in general, and the Society's educational activities in particular, that the various activities of the Math Competitons Committee are running so smoothly.

Daryl Tingley, Chair
Mathmatical Competitions Committee

The 34th (2002) Canadian Mathematical Olympiad was held on Wednesday, March 27th, 2002 with 80 competitors from 47 schools in eight Canadian provinces participating. The number of contestants from each province was as follows:

BC (12) AB (9) MB (2) ON (47) QC (5) NB (2) NS (2) NF (1)

The contest paper consisted of five questions. Each question was worth 7 marks for a total maximum score of $m = 35$. The contestants performances were grouped into four divisions as follows.

Division	**Total Score**	**No. of Students**
I	$28 \leq m \leq 35$	10
II	$20 \leq m < 28$	12
III	$15 \leq m < 20$	17
IV	$0 \leq m < 15$	41

PRIZE WINNERS

First prize:

Tianyi Han, Woburn Collegiate Institute,ON
The Sun Life Cup, $2000, and a book prize

Second prize:

Roger Mong, Don Mills Collegiate Institute,ON
$1500 and a book prize

Third prize:

Paul Cheng, West Vancouver Secondary School,BC
$1000 and a book prize

Honourable Mentions:

Robert Barrington Leigh, Old Scona Academy,AB
Olena Bormashenko, Don Mills Collegiate Institute,ON
Xiaoxuan Jin, Vincent Massey Secondary School,ON
Timothy Kusalik, Queen Elizabeth High School,NS
Cornwall Lau, David Thompson Secondary School,BC
Feng Tian, Vincent Massey Secondary School,ON
Yang Yang, Don Mills Collegiate Institute,ON
$500 and a book prize

SECOND DIVISION

Score: $20 \leq m < 28$

Brian Choi	Markville Secondary School,ON
Kevin Chung	Earl Haig Secondary School,ON
Shih En Lu	Marianopolis College,QC
Sumudu Fernando	Harry Ainlay Comp. High School,AB
Alex Fink	Queen Elizabeth High School,AB
Ralph Furmaniak	Tom Griffiths Home School,ON
Robin Li	St. Patrick Secondary School,ON
Henry Pan	East York Collegiate Institute,ON
Louis Francois Preville Ratel	College de l'Assomption,QC
Yin Ren	Vincent Massey Secondary School,ON
Jacob Tsimerman	Univ. of Toronto School,ON

THIRD DIVISION

Score: $15 \leq m < 20$

Maximilian Butler	Tom Griffiths Home School,ON
Aaron Chan	J.N. Burnett Secondary School,BC
Peter Du	Sir Winston Churchill High School,AB
Fan Feng	Vincent Massey Secondary School,ON
Ryan Holm	St. Ignatius High School,ON
Liang Hong	Univ. of Toronto Schools,ON
Oleg Ivrii	Don Mills Collegiate Institute,ON
Jenny Yue Jin	Earl Haig Secondary School,ON
Keigo Kawaji	Earl Haig Secondary School,ON
Songhao Li	L'Amoureaux Collegiate Institute,ON
Yichaun Liu	University Hill Secondary School,BC
Andrew Mao	Tom Griffiths Home School,ON
Mathieu Guay Paquet	Secondaire Antoine Brossard,QC
Alex Shyr	VSB/UBC Transition Program,BC
Sarah Sun	St. Mary's School,AB
Lin Ray Wung	Pinetree Secondary School,BC
David (Xin) Zhang	Woburn Collegiate Institute,ON

FOURTH DIVISION

Score: $0 \leq m < 15$

Robert Biswas	Vincent Massey Secondary School,ON
Tiffany Chao	Sir Winston Churchill Secondary,BC
Valerie Cheung	Vincent Massey Secondary School,ON
Leonid Chindelevitch	Marianopolis College,QC
Kevin Choi	Crescent School,ON
Keith Chung	Western Canada High School,AB
Mark Daniels	Comm. Hebrew Academy of Toronto,ON

Rowan Dorin	St. John's-Ravenscourt School,MB
Jerome Grand Maison	CEGEP de la Gaspesie,QC
Michael Hirsch	St. John's-Ravenscourt School,MB
Jason Hornosty	Fredericton High School,NB
Marina Hu	Burnaby South Secondary School,BC
Pawel Kosicki	Vincent Massey Secondary School,ON
Janos Kramar	Univ. of Toronto Schools,ON
Hyon Lee	Vincent Massey Secondary School,ON
Vincent Leung	Upper Canada College,ON
Charles Li	Western Canada High School,AB
Angela Lin	Sir Winston Churchill Secondary,BC
Mike Liu	Waterloo Collegiate Institute,ON
Micah McCurdy	Saint Patrick's High School,NS
Marcin Mika	Father Michael Goetz Secondary,ON
Alec Mills	Western Canada High School,AB
Amit Mukerji	Vincent Massey Secondary School,ON
Jiafei Niu	Waterloo Collegiate Institute,ON
Avery Owen	Don Mills Collegiate Institute,ON
Sharon Shao	Eric Hamber Secondary School,BC
Yihao Shen	Saint John High School,NB
Christopher Tam	Upper Canada College,ON
Alvin Tan	McNally Comp. High School,AB
Leonid Tchourakov	Grand River Collegiate,ON
Samuel Wong	University Hill Secondary School,BC
Nithum Thain	Prince of Wales Collegiate,NF
Wei Lung Tseng	Yale Secondary School,BC
Yves Wang	Northern Secondary School,ON
Chris Woo	Crescent School,ON
Kevin Yip	Don Mills Collegiate Institute,ON
Dongbo Yu	Don Mills Collegiate Institute,ON
Matei Zaharia	Jarvis Collegiate Institute,ON
Dapeng Zhao	Vincent Massey Secondary School,ON
Yin Zhao	Vincent Massey Secondary School,ON
Anjie Zhou	Westdale Secondary School,ON
Zhongying Zhou	Vincent Massey,ON

THE THIRTY FOURTH
CANADIAN MATHEMATICAL OLYMPIAD
2002

1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

   What is the maximum number of elements that S can contain?

2. Call a positive integer n **practical** if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

   For example, the divisors of 6 are **1**, **2**, **3**, and **6** . Since

   $$1=\mathbf{1},\quad 2=\mathbf{2},\quad 3=\mathbf{3},\quad 4=\mathbf{1}+\mathbf{3},\quad 5=\mathbf{2}+\mathbf{3},\quad 6=\mathbf{6},$$

   we see that 6 is practical.

   Prove that the product of two practical numbers is also practical.

3. Prove that for all positive real numbers a, b, and c,

   $$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c,$$

   and determine when equality occurs.

4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q.

   Prove that $PQ = r$.

5. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

   $$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

   for all x and y in $\mathbb{N}$.

1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

   What is the maximum number of elements that S can contain?

   Solution 1
   It can be checked that all the sums of pairs for the set $\{1, 2, 3, 5, 8\}$ are different.

   Suppose, for a contradiction, that S is a subset of $\{1, \ldots, 9\}$ containing 6 elements such that all the sums of pairs are different. Now the smallest possible sum for two numbers from S is $1 + 2 = 3$ and the largest possible sum is $8 + 9 = 17$. That gives 15 possible sums: $3, \ldots, 17$. Also there are $\binom{6}{2} = 15$ pairs from S. Thus, each of $3, \ldots, 17$ is the sum of exactly one pair. The only pair from $\{1, \ldots, 9\}$ that adds to 3 is $\{1, 2\}$ and to 17 is $\{8, 9\}$. Thus $1, 2, 8, 9$ are in S. But then $1+9 = 2+8$, giving a contradiction. It follows that the maximum number of elements that S can contain is 5.

   Solution 2.

   It can be checked that all the sums of pairs for the set $\{1, 2, 3, 5, 8\}$ are different.

   Suppose, for a contradiction, that S is a subset of $\{1, \ldots 9\}$ such that all the sums of pairs are different and that $a_1 < a_2 < \ldots < a_6$ are the members of S.

   Since $a_1 + a_6 \neq a_2 + a_5$, it follows that $a_6 - a_5 \neq a_2 - a_1$. Similarly $a_6 - a_5 \neq a_4 - a_3$ and $a_4 - a_3 \neq a_2 - a_1$. These three differences must be distinct positive integers, so,

$$(a_6 - a_5) + (a_4 - a_3) + (a_2 - a_1) \geq 1 + 2 + 3 = 6\,.$$

   Similarly $a_3 - a_2 \neq a_5 - a_4$, so

$$(a_3 - a_2) + (a_5 - a_4) \geq 1 + 2 = 3\,.$$

   Adding the above 2 inequalities yields

$$a_6 - a_5 + a_5 - a_4 + a_4 - a_3 + a_3 - a_2 + a_2 - a_1 \geq 6 + 3 = 9\,,$$

   and hence $a_6 - a_1 \geq 9$. This is impossible since the numbers in S are between 1 and 9.

2. Call a positive integer n *practical* if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

   For example, the divisors of 6 are **1**, **2**, **3**, and **6** . Since
   1=**1**, 2=**2**, 3=**3**, 4=**1**+**3**, 5=**2**+ **3**, 6=**6**,

   we see that 6 is practical.

   Prove that the product of two practical numbers is also practical.

   Solution
   Let p and q be practical. For any $k \leq pq$, we can write

$$k = aq + b \text{ with } 0 \leq a \leq p,\ 0 \leq b < q.$$

   Since p and q are practical, we can write

$$a = c_1 + \ldots + c_m, \quad b = d_1 + \ldots + d_n$$

   where the c_i's are distinct divisors of p and the d_j's are distinct divisors of q. Now

$$\begin{aligned} k &= (c_1 + \ldots + c_m)q + (d_1 + \ldots + d_n) \\ &= c_1 q + \ldots + c_m q + d_1 + \ldots + d_n. \end{aligned}$$

   Each of $c_i q$ and d_j divides pq. Since $d_j < q \leq c_i q$ for any i, j, the $c_i q$'s and d_j's are all distinct, and we conclude that pq is practical.

3. Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\geq a+b+c,$$

and determine when equality occurs.

Each of the inequalities used in the solutions below has the property that equality holds if and only if $a=b=c$. Thus equality holds for the given inequality if and only if $a=b=c$.

Solution 1.

Note that $a^4+b^4+c^4=\frac{(a^4+b^4)}{2}+\frac{(b^4+c^4)}{2}+\frac{(c^4+a^4)}{2}$. Applying the arithmetic-geometric mean inequality to each term, we see that the right side is greater than or equal to

$$a^2b^2+b^2c^2+c^2a^2.$$

We can rewrite this as

$$\frac{a^2(b^2+c^2)}{2}+\frac{b^2(c^2+a^2)}{2}+\frac{c^2(a^2+b^2)}{2}.$$

Applying the arithmetic mean-geometric mean inequality again we obtain $a^4+b^4+c^4\geq a^2bc+b^2ca+c^2ab$. Dividing both sides by abc (which is positive) the result follows.

Solution 2.

Notice the inequality is homogeneous. That is, if a,b,c are replaced by ka,kb,kc, $k>0$ we get the original inequality. Thus we can assume, without loss of generality, that $abc=1$. Then

$$\begin{aligned}\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} &= abc\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right)\\ &= a^4+b^4+c^4.\end{aligned}$$

So we need prove that $a^4+b^4+c^4\geq a+b+c$.

By the Power Mean Inequality,

$$\frac{a^4+b^4+c^4}{3}\geq\left(\frac{a+b+c}{3}\right)^4,$$

so $a^4+b^4+c^4\geq(a+b+c)\cdot\frac{(a+b+c)^3}{27}$.

By the arithmetic mean-geometric mean inequality, $\frac{a+b+c}{3}\geq\sqrt[3]{abc}=1$, so $a+b+c\geq 3$.

Hence, $a^4+b^4+c^4\geq(a+b+c)\cdot\frac{(a+b+c)^3}{27}\geq(a+b+c)\frac{3^3}{27}=a+b+c$.

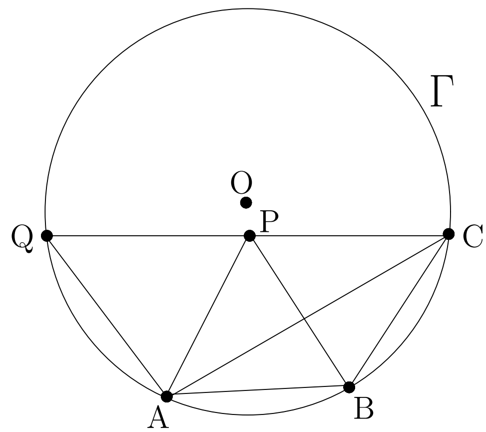
Solution 3.

Rather than using the Power-Mean inequality to prove $a^4+b^4+c^4\geq a+b+c$ in Proof 2, the Cauchy-Schwartz-Bunjakovsky inequality can be used twice:

$$\begin{aligned}(a^4+b^4+c^4)(1^2+1^2+1^2) &\geq (a^2+b^2+c^2)^2\\ (a^2+b^2+c^2)(1^2+1^2+1^2) &\geq (a+b+c)^2\end{aligned}$$

So $\frac{a^4+b^4+c^4}{3}\geq\frac{(a^2+b^2+c^2)^2}{9}\geq\frac{(a+b+c)^4}{81}$. Continue as in Proof 2.

4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let CP meet Γ again at Q. Prove that $PQ = r$.



Solution 1.

Let the center of Γ be O, the radius r. Since $BP = BC$, let $\theta = \measuredangle BPC = \measuredangle BCP$.

Quadrilateral $QABC$ is cyclic, so $\measuredangle BAQ = 180° - \theta$ and hence $\measuredangle PAQ = 120° - \theta$.

Also $\measuredangle APQ = 180° - \measuredangle APB - \measuredangle BPC = 120° - \theta$, so $PQ = AQ$ and $\measuredangle AQP = 2\theta - 60°$.

Again because quadrilateral $QABC$ is cyclic, $\measuredangle ABC = 180° - \measuredangle AQC = 240° - 2\theta$.

Triangles OAB and OCB are congruent, since $OA = OB = OC = r$ and $AB = BC$.

Thus $\measuredangle ABO = \measuredangle CBO = \frac{1}{2}\measuredangle ABC = 120° - \theta$.

We have now shown that in triangles AQP and AOB, $\measuredangle PAQ = \measuredangle BAO = \measuredangle APQ = \measuredangle ABO$. Also $AP = AB$, so $\triangle AQP \cong \triangle AOB$. Hence $QP = OB = r$.

Solution 2.

Let the center of Γ be O, the radius r. Since A, P and C lie on a circle centered at B, $60° = \measuredangle ABP = 2\measuredangle ACP$, so $\measuredangle ACP = \measuredangle ACQ = 30°$.

Since Q, A, and C lie on Γ, $\measuredangle QOA = 2\measuredangle QCA = 60°$.

So $QA = r$ since if a chord of a circle subtends an angle of $60°$ at the center, its length is the radius of the circle.

Now $BP = BC$, so $\measuredangle BPC = \measuredangle BCP = \measuredangle ACB + 30°$.

Thus $\measuredangle APQ = 180° - \measuredangle APB - \measuredangle BPC = 90° - \measuredangle ACB$.

Since Q, A, B and C lie on Γ and $AB = BC$, $\measuredangle AQP = \measuredangle AQC = \measuredangle AQB + \measuredangle BQC = 2\measuredangle ACB$. Finally, $\measuredangle QAP = 180 - \measuredangle AQP - \measuredangle APQ = 90 - \measuredangle ACB$.

So $\measuredangle PAQ = \measuredangle APQ$ hence $PQ = AQ = r$.

5. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.

Solution 1.

We claim that f is a constant function. Suppose, for a contradiction, that there exist x and y with $f(x) < f(y)$; choose x, y such that $f(y) - f(x) > 0$ is minimal. Then

$$f(x) = \frac{xf(x) + yf(x)}{x + y} < \frac{xf(y) + yf(x)}{x + y} < \frac{xf(y) + yf(y)}{x + y} = f(y)$$

so $f(x) < f(x^2 + y^2) < f(y)$ and $0 < f(x^2 + y^2) - f(x) < f(y) - f(x)$, contradicting the choice of x and y. Thus, f is a constant function. Since $f(0)$ is in $\mathbb{N}$, the constant must be from $\mathbb{N}$.

Also, for any c in $\mathbb{N}$, $xc + yc = (x + y)c$ for all x and y, so $f(x) = c,\ c \in \mathbb{N}$ are the solutions to the equation.

Solution 2.

We claim f is a constant function. Define $g(x) = f(x) - f(0)$. Then $g(0) = 0,\ g(x) \geq -f(0)$ and

$$xg(y) + yg(x) = (x + y)g(x^2 + y^2)$$

for all x, y in $\mathbb{N}$.

Letting $y = 0$ shows $g(x^2) = 0$ (in particular, $g(1) = g(4) = 0$), and letting $x = y = 1$ shows $g(2) = 0$. Also, if x, y and z in $\mathbb{N}$ satisfy $x^2 + y^2 = z^2$, then

$$g(y) = -\frac{y}{x}g(x). \qquad (*)$$

Letting $x = 4$ and $y = 3$, $(*)$ shows that $g(3) = 0$.

For any even number $x = 2n > 4$, let $y = n^2 - 1$. Then $y > x$ and $x^2 + y^2 = (n^2 + 1)^2$. For any odd number $x = 2n+1 > 3$, let $y = 2(n+1)n$. Then $y > x$ and $x^2 + y^2 = ((n+1)^2 + n^2)^2$. Thus for every $x > 4$ there is $y > x$ such that $(*)$ is satisfied.

Suppose for a contradiction, that there is $x > 4$ with $g(x) > 0$. Then we can construct a sequence $x = x_0 < x_1 < x_2 < \ldots$ where $g(x_{i+1}) = -\dfrac{x_{i+1}}{x_i}\, g(x_i)$. It follows that $|g(x_{i+1})| > |g(x_i)|$ and the signs of $g(x_i)$ alternate. Since $g(x)$ is always an integer, $|g(x_{i+1})| \geq |g(x_i)| + 1$. Thus for some sufficiently large value of i, $\ g(x_i) < -f(0)$, a contradiction.

As for Proof 1, we now conclude that the functions that satisfy the given functional equation are $f(x) = c,\ c \in \mathbb{N}$.

Solution 3. Suppose that W is the set of nonnegative integers and that $f : W \to W$ satisfies:

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2). \qquad (*)$$

We will show that f is a constant function.

Let $f(0) = k$, and set $S = \{x \mid f(x) = k\}$.

Letting $y = 0$ in $(*)$ shows that $f(x^2) = k \ \ \forall\ x > 0$, and so

$$x^2 \in S \ \ \forall\ x > 0 \qquad (1)$$

In particular, $1 \in S$.

Suppose $x^2 + y^2 = z^2$. Then $yf(x) + xf(y) = (x+y)f(z^2) = (x+y)k$. Thus,

$$x \in S \quad \text{iff} \quad y \in S. \tag{2}$$

whenever $x^2 + y^2$ is a perfect square.

For a contradiction, let n be the smallest non-negative integer such that $f(2^n) \neq k$. By (l) n must be odd, so $\dfrac{n-1}{2}$ is an integer. Now $\dfrac{n-1}{2} < n$ so $f(2^{\frac{n-1}{2}}) = k$. Letting $x = y = 2^{\frac{n-1}{2}}$ in $(*)$ shows $f(2^n) = k$, a contradiction. Thus every power of 2 is an element of S.

For each integer $n \geq 2$ define $p(n)$ to be the *largest prime* such that $p(n) \mid n$.

Claim: For any integer $n > 1$ that is not a power of 2, there exists a sequence of integers $x_1, x_2, \ldots, x_r$ such that the following conditions hold:

a) $x_1 = n$.

b) $x_i^2 + x_{i+1}^2$ is a perfect square for each $i = 1, 2, 3, \ldots, r-1$.

c) $p(x_1) \geq p(x_2) \geq \ldots \geq p(x_r) = 2$.

Proof: Since n is not a power of 2, $p(n) = p(x_1) \geq 3$. Let $p(x_1) = 2m+1$, so $n = x_1 = b(2m+1)^a$, for some a and b, where $p(b) < 2m+1$.

Case 1: $a = 1$. Since $(2m+1, 2m^2+2m, 2m^2+2m+1)$ is a Pythagorean Triple, if $x_2 = b(2m^2 + 2m)$, then $x_1^2 + x_2^2 = b^2(2m^2+2m+1)^2$ is a perfect square. Furthermore, $x_2 = 2bm(m+1)$, and so $p(x_2) < 2m+1 = p(x_1)$.

Case 2: $a > 1$. If $n = x_1 = (2m+1)^a \cdot b$, let $x_2 = (2m+1)^{a-1} \cdot b \cdot (2m^2+2m)$, $x_3 = (2m+1)^{a-2} \cdot b \cdot (2m^2+2m)^2$, …, $x_{a+1} = (2m+1)^0 \cdot b \cdot (2m^2+2m)^a = b \cdot 2^a m^a (m+1)^a$. Note that for $1 \leq i \leq a$, $x_i^2 + x_{i+1}^2$ is a perfect square and also note that $p(x_{a+1}) < 2m+1 = p(x_1)$.

If x_{a+1} is not a power of 2, we extend the sequence x_i using the same procedure described above. We keep doing this until $p(x_r) = 2$, for some integer r.

By (2), $x_i \in S$ iff $x_{i+1} \in S$ for $i = 1, 2, 3, \ldots, r-1$. Thus, $n = x_1 \in S$ iff $x_r \in S$. But x_r is a power of 2 because $p(x_r) = 2$, and we earlier proved that powers of 2 are in S. Therefore, $n \in S$, proving the claim.

We have proven that every integer $n \geq 1$ is an element of S, and so we have proven that $f(n) = k = f(0)$, for each $n \geq 1$. Therefore, f is constant, Q.E.D.

GRADERS' REPORT

Each question was worth a maximum of 7 marks. Every solution on every paper was graded by two different markers. If the two marks differed by more than one point, the solution was reconsidered until the difference was resolved. If the two marks differed by one point, the average was used in computing the total score. The top papers were then reconsidered until the committee was confident that the prize winning contestants were ranked correctly.

The various marks assigned to each solution are displayed below, as a percentage.

MARKS	#1	#2	#3	#4	#5
0	1.2	28.1	31.9	51.9	37.5
1	2.5	10.6	11.9	13.8	24.4
2	10.6	12.5	10.0	3.1	13.8
3	6.9	12.5	1.9	3.1	11.9
4	12.5	6.2	0.0	0.6	5.0
5	8.1	3.1	1.9	0.6	0.0
6	6.9	3.1	9.4	0.6	1.9
7	51.2	23.8	33.1	26.2	5.6

PROBLEM 1

Two points were awarded for an example of a set S with 5 elements. Most students realized that they should then assume that S contained 6 elements and derive a contradiction.

Contestants who listed all mutually exclusive pairs of numbers (eg S cannot contain both the sets $\{1,3\}$ and $\{8,9\}$) wasted a lot of time, with little or no progress.

Successful contestants realized that they must concentrate on possible differences between elements of the set. Unfortunately, several made unnecessary assumptions along these lines: "Place 1 in the set S. If S contains $a_1, \ldots, a_k$, then add a_{k+1} so that the difference $a_{k+1} - a_k$ is as small as possible", without any justification that such an approach would find the largest possible set S.

These mistakes aside, there were many well-written solutions. Proof 1 of the Official Solutions was most common. Students who used a proof like Proof 2 wrote very nice solutions. Another successful approach was to derive restrictions on possible values of differences between adjacent numbers in the ordered set S. Several of these proofs lost points for poor explanation.

PROBLEM 2

The only solution to this problem found by either the contestants or the committee is that of the Official solutions. Referring to the Official Solutions, the key to the problem is to write $k = ap + b$ with $0 \le a \le p$ and $0 \le b < q$. Almost all contestants who thought of this went on to correctly solve the problem.

Many contestants considered the cases $0 < k \le p, p < k \le 2p, 2p < k \le 3p, \cdots$. Although this is in effect using $k = ap + b$, contestants who proceeded in this fashion usually failed to show that the sum consisted of distinct factors, so were awarded 2 or 3 points.

One point was awarded to contestants who noticed something interesting about practical numbers, such as practical numbers (other than 1) must be even, or powers of 2 are practical.

PROBLEM 3

This problem can be done in a multitude of ways, as evidenced by the fact that over ten different correct solutions were presented. Most of the correct solutions used some variation of the AM-GM inequality or the Cauchy-Schwartz inequality. However, the problem can be solved without knowing these tools; one can do some clever algebraic manipulations to produce the correct result. Students received 1 or 2 points for some non-trivial work with the inequalities, and could receive further partial credit if their work was leading to a solution.

PROBLEM 4

Many distinct solutions were submitted, but all correct solutions used classical geometry and/or trigonometry. No contestants were successful in using coordinates, and no contestants attempted a transformation based approach. One point was awarded for finding a non-trivial relation between angles or for the special case $AD = r$. Two points were awarded for finding $QA = QP$. Three points were awarded for finding that $\angle QOA = 60°$. Contestants did not receive more than three points unless they made substantial progress towards a solution.

PROBLEM 5

Few students made significant progress on this extremely challenging problem. The correct answer is that $f(x)$ equals some constant c, for all non-negative integers x. Students received one point for verifying that $f(x) = c$ is a solution. Some students were able to verify the result when x is a perfect square or when x is a power of 2, for which they received a further point. Only six students attained a mark of 6 or 7 for this problem: four of these solutions were very similar to the first Official Solution, and the others were a combination of the second and third Official Solutions.