Exact and Approximate Methods for Nonlinear Differential Equations
Méthodes exactes et approximatives pour la résolution des équations différentielles non-linéaires
(Org: **Alexei F. Cheviakov** and/et **George W. Patrick** (Saskatchewan))

STEPHEN ANCO, Department of Mathematics, Brock University
Conserved integrals of compressible fluid flow in $n > 1$ spatial dimensions

I will present a summary of recent work on conservation laws of compressible fluid flow in $n > 1$ spatial dimensions. This work applies the general method of Euler operators and multipliers to give a complete classification of local conservation laws and conserved integrals for two primary cases of physical and mathematical interest in the study of fluid flow:

(1) kinematic conservation laws, like mass, energy, momentum and angular momentum, for which the conserved density and flux depend only on the fluid velocity, pressure and density (but not their spatial derivatives), in addition to the time and space coordinates;

(2) vorticity conservation laws, such as three-dimensional helicity and two-dimensional circulation, where the conserved density and flux have an essential dependence on the curl of the fluid velocity in a form exhibiting odd parity under spatial reflections.

In particular, all kinematic and vorticity conservation laws holding for special equations of state or in special dimensions are explicitly obtained.

ALEXEI CHEVIAKOV, University of Saskatchewan, Saskatoon, Canada
Narrow escape problems: asymptotic optimization of small trap locations for the unit sphere

We first consider the motion of a Brownian particle in a sphere with an almost entirely reflecting boundary, containing small absorbing windows. In the limit of asymptotically small total window area, we compute three leading terms of the mean first passage time (MFPT), the average time required for a randomly placed Brownian particle to escape from the domain. The third term of the asymptotic expansion depends on relative window locations. We study this dependence and numerically compute configurations of N identical windows which minimize the MFPT, for $2 \leq N \leq 20$.

A related problem with totally reflecting boundary and *interior* spherical traps of asymptotically small radii is also considered. Three terms of asymptotic expansion of the MFPT are computed, and optimal N-trap volume configurations are found for $2 \leq N \leq 20$.

It is also shown that the problem of asymptotic minimization of MFPT, for both of the above set-ups, is equivalent to the problem of maximization of the principal eigenvalue of the Laplacian in each respective domain.

THEODORE KOLOKOLNIKOV, Dalhousie
Simple PDE model of spot replication in any dimension

We propose a simple PDE model which exhibits self-replication of spot solutions in any dimension. This model is analysed in one and higher dimensions. In one dimension, we rigorously demonstrate that the conditions proposed by Nishiura and Ueyama for self-replication are satisfied. In dimension three, two different types of replication mechanisms are analysed. The first type is due to radially symmetric instability, whereby a spot bifurcates into a ring. The second type is non-radial instability, which causes a spot to deform into a peanut-like shape, and eventually split into two spots. Both types of replication are observed in our model, depending on parameter choice. Numerical simulations are shown confirming our analytical results.

This is a joint work with Chiun-Chuan Chen.

NILOOFAR MANI, University of Western Ontario, London, Ontario
Fast numeric geometric techniques for computer generated DAE models

This talk is the second of two talks in the session (the first being that of Greg Reid) which focuses on numeric-geometric algorithms for nonlinear systems of ODE with constraints.

Such so-called DAE arise frequently in applications, and are often so complicated that they are automatically generated by computer environments (in our case, the MapleSim environment). Missing constraints arising by prolongation (differentiation) of the DAE need to be determined to consistently initialize and stabilize their numerical solution. In this talk, we review a fast prolongation method to include constraints. Our symbolic-numeric method avoids the unstable eliminations of exact approaches. The method is successful provided the prolongations with respect to a single dominant independent variable have a block structure which is efficiently uncovered by Linear Programming.

Constrained mechanical systems generated by MapleSim are used to demonstrate the power of the approach. Using the fast prolongation method and block structure give us missing constraints and initial condition respectively for the system. Globally qualitatively distinct classes of solutions are determined by using Bertini software, whose approach is global homotopy continuation method based on numerical algebraic geometry. Finally, we built efficient interface to Maple's dsolve/numeric for these different models.

MARYAM NAMAZI, University of Victora, Victoria, BC, Canada
A pitfall in using central scheme in atmospheric modeling

Hyperbolic conservation laws are widely used in applied sciences and engineering, such as fluid mechanics, biology and environmental science. However due the inherent complex non-homogeneities and nonlinearities, they are not tractable analytically, but instead, various numerical methods have been proposed for their resolution.

The so-called finite volume methods for conservation laws, that are the state-of-the-art, can be classified into two categories: Godunov-type methods that rely on the detailed or approximate solution of a Riemann problem at each cell or control-volume interface, where the numerical solution is made discontinuous, and central schemes that avoid the Riemann problem all together by averaging between neighboring cells to yield a smooth solution at the cell interfaces and move the discontinuity to the cell center. While central schemes are very attractive because they are very cheap to run and easy to implement, their 2D version can lead to very misleading results when used to simulate wave phenomena, for long integration times. The smoothing of the solution in the direction perpendicular to the direction of the wave-propagation distorts significantly the dispersion characteristics of the waves and induces important deformations in the solution shape and propagation speed.

In this talk, we present such behavior of the central scheme for the case of equatorially trapped waves that are an important component in the dynamics of the tropical climate and the large-scale atmospheric circulation. We provide numerical examples supplemented with a harmonic analysis for the frozen stencil to demonstrate the phenomenon and backup our claims. In order to avoid such disasters we suggest, for such wave problems, use the one-dimensional central scheme compounded through directional splitting.

GEORGE PATRICK, University of Saskatchewan
On converting any one-step method to a variational integrator of the same order

In the formalism of constrained mechanics, such as that which underlies the Shake and Rattle methods of molecular dynamics, we present an algorithm to convert any one-step integration method to a variational integrator of the same order. The one-step method is arbitrary, and the conversion can be automated, resulting in a powerful and flexible approach to the generation of novel variational integrators.

GREG REID, University of Western Ontario, London, Ontario
Stable numerical-geometric approaches for differential algebraic equations

This is the first of two talks about stable numeric-geometric methods for general systems of differential equations with constraints (so-called differential-algebraic equations or DAE). Such systems are attracting much attention since they are automatically generated by computer modeling environments such as MapleSim. Determination of such constraints is essential for the determination of consistent initial conditions and the numerical solution of such systems. This talk will concentrate on introduction of concepts from the (Jet) geometry of differential equations, illustrated by visualizations and simple examples. A subsequent talk by Niloofar Mani, will discuss initial investigations that we have made using MapleSim, and such approaches.

This talk will be an introduction to stable numerical methods for such general systems. The corresponding problem for the non-differential case, that of approximate polynomial systems, has only recently been given a solution, through the works of Sommese, Wampler and others. The new area called numerical algebraic geometry, will also be described. Key data structures are certain witness points on jet manifolds of solutions, computed by stable homotopy continuation methods.

ARTUR SOWA, University of Saskatchewan
On the topological characteristic of the spectra of some composite quantum systems

I will discuss the spectral characteristic of a class of composite quantum systems that obey nonlinear and nonlocal type dynamics. The specific models of dynamics I consider depend on a triplet of constituents—two subsystem Hamiltonians $(H, \hat{H})$, both with pure point spectrum, and an analytic function (f)—as well as a real parameter s. The composite system state vector (canonically identified with a Hilbert–Schmidt class operator, here denoted K) evolves according to the nonlinear equation:

$$-i\hbar\dot{K} = KH + \hat{H}K + \frac{1}{s}Kf(K^*K).$$

In spite of the complex patterns of the resulting dynamics, the *stationary states* of such systems can be described fairly explicitly (see A. Sowa, *Stationary states in nonlocal type dynamics of composite systems*, J. Geom. Phys., to appear, doi:10.1016/j.geomphys.2009.07.015). Here, I will present recent results characterizing the topological type of the spectra of such composite systems.

It turns out that the spectrum of the composite system is often very distinct from the spectra of its two subsystems. I will illustrate this with examples which show that while subsystems have discrete spectra, the energy levels of the composite system may fill the Cantor set, or, in another case, a union of a finite number of intervals. Some of the examples involve the series $\sum n^{-s}$.

CRISTINA STOICA, Wilfrid Laurier University, 75 Univ. Ave West, Waterloo, ON N2L 3C5
Escape dynamics in a collinear three-point-mass system

We present studies concerning the escape mechanism in collinear three-point-mass systems with small-range-repulsive/large-range-attractive pairwise interaction for non-negative energies.

We show that for zero energy, the set initial conditions leading to escape configurations where all three separations infinitely increase, has zero Lebesque measure. Also, in the case of mass-symmetric systems, we give numerical evidence of the existence of a periodic orbit that is reminiscent of the von Schubart orbit in the collinear three body problem of celestial mechanics. For positive energies, we prove that the set of initial conditions leading to escape configurations has positive Lebesque measure.

This is joint work with Manuele Santoprete (Wilfrid Laurier) and Danile Pasca (U Oradea, Romania).

JACEK SZMIGIELSKI, University of Saskatchewan, Department of Mathematics and Statistics, Wiggins 106, Saskatoon, S7N 5E6
Orthogonal and Biorthogonal Polynomials and integrable models of weakly dispersive water waves

One of the most studied in recent years nonlinear PDEs is the Camassa–Holm equation (CH) and its descendant, Degasperis–Procesi equation (DP), as well as a few other equations discovered within last year. One feature of these equations that stands out is that they possess interesting distributional (weak) solutions, non-smooth in a classical sense. Among them, the non-smooth solitons (peakons), first discovered by Camassa and Holm, exhibit remarkable mathematical features, which can be traced back to classical problems in the theory of orthogonal polynomials, essentially going back to T. Stieltjes. This connection becomes highly nontrivial for the case of the DP equation, for which the underlying polynomials belong to a new family of biorthogonal polynomials associated to the Cauchy kernel $1/(x+y)$. Fundamental properties of these biorthogonal polynomials were studied by M. Bertola, M. Gekthman and J. S. These polynomials can be used to construct explicit peakon solutions for the cubic version of the CH equation, the VN equation, discovered by V. Novikov. The last part of the talk gives an alternative construction of peakons for the VN equation given in the joint paper by A. Hone, H. Lundmark and J. S. (Dynamics of PDE, Vol.6, No.3, 253–289, 2009).