Variational Methods in Partial Differential Equations

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LIA BRONSARD, McMaster University, Hamilton, ON, Canada

Vortices for a rotating toroidal Bose–Einstein condensate

We construct local minimizers of the Gross–Pitaevskii energy, introduced to model Bose–Einstein condensates (BEC) in the Thomas–Fermi regime which are subject to a uniform rotation. Our sample domain is taken to be a solid torus of revolution in $\mathbf{R}^3$ with starshaped cross-section. We show that for angular speeds $\omega_\epsilon = O(|\ln \epsilon|)$ there exist local minimizers of the energy which exhibit vortices, for small enough values of the parameter ϵ. These vortices concentrate at one or several planar arcs (represented by integer multiplicity rectifiable currents) which minimize a line energy, obtained as a Γ-limit of the Gross–Pitaevskii functional. The location of these limiting vortex lines can be described under certain geometrical hypotheses on the cross-sections of the torus.

These are results obtained in collaboration with S. Alama and J. A. Montero.

RUSTUM CHOKSI, Simon Fraser University

Scaling laws during the onset and destruction of the intermediate state in a type-I superconductor

The intermediate state of type-I superconductors is a classical pattern-formation problems in physics, first studied by Landau in 1937. Here we explore the ground state energy from the point of view of rigorous scaling laws. We find precisely five parameter regimes each associated with an optimal construction and scaling law, thereby proving that exactly those five different regimes are traversed with increasing magnetic field.

This is joint work with Sergio Conti (Duisburg–Essen), Bob Kohn (Courant) and Felix Otto (Bonn).

WALTER CRAIG, McMaster University, Hamilton, ON, Canada

Remarks on the singular set of solutions of the Navier–Stokes equations

This presentation will discuss several results on the space-time set of singlarities of (energy inequality satisfying) weak solutions of the Navier–Stokes equations.

This is recent joint work with A. Biryuk and S. Ibrahim.

IVAR EKELAND, University of British Columbia

A new type of differential equation arising from economic theory

In optimal control, the discount rate is always exponential, that is, a gain of u occurring at a distance (in time) t from now is worth a gain $u \exp(-rt)$ today, where $r > 0$ is the interest rate. Using this expression, one derives the classical Hamilton–Jacobi–Bellman equation.

In economics, there is no reason to favour exponential discount rates. Much interest recently has been paid to discount rates $h(t)$, where $h(0) = 1$, h is decreasing and $h(t)$ goes to zero when t goes to infinity. With such a discount rate, the optimal control loses economic significance, and must be replaced by an equilibrium strategy. The latter is given by a new equation, which resembles the HJB equation, but which is no longer a PDE.

RENATO ITURRIAGA, CIMAT

Physical solutions of the Hamilton–Jacobi equation

We consider a Lagrangian system on the d-dimensional torus, and the associated Hamilton–Jacobi equation. Assuming that the Aubry set of the system consists in a finite number of hyperbolic periodic orbits of the Euler–Lagrange flow, we study the vanishing-viscosity limit, from the viscous equation to the inviscid problem. Under suitable assumptions, we show that solutions of the viscous Hamilton–Jacobi equation converge to a unique solution of the inviscid problem.

HECTOR LOMELI, Instituto Tecnológico Autónomo de México ITAM, Río Hondo #1, México DF 01000

Invariant manifolds, variational principles and dynamic programming

The optimality principle of Bellman is frequently used to solve problems in dynamic programming. The optimal selection of the dynamic control is the optimal policy. The method of Bellman leads to the so-called Hamilton–Jacobi–Bellman PDE.

An important observation is that there are areas of dynamics that use variational methods similar to the one of Bellman. In particular, it is possible to use a variational principle to approximate and study the stable and unstable manifolds of a saddle fixed point. In this work we explore the dynamic properties of the principle of Bellman.

ANTONMARIA MINZONI, UNAM, Ciudad Universitaria, Mexico DF

Stability of embedded solitons at the edge of the continuum

We consider the problem of two hump solutions of the modified NLS equation which describes short optical pulses.

Multihump solutions are obtained asymptotically using a modulation formulation on the Lagrangian coupled to a free boundary for the radiation. We study the one sided stability using the modulation coupled with the radiation.

The effects which produce the multiple humps and their instabilty are exponentially small in the distance between humps.

We show that the asymptotic theory explains completely the numerics in the dynamical evolution.

We commment on the possibility of making rigorous this asymptotic theory.

ALBERTO MONTERO ZARATE, University of Toronto

A Gamma convergence result for the Gross Pitaevskii energy in $\mathbb{R}^3$

The Gross Pitaevskii energy is a functional often used to model Bose Einstein condensates trapped in a potential. We consider this energy in all of $\mathbb{R}^3$, under a mass constraint, and find its gamma limit as a certain parameter in the energy goes to infinity. Among other things this requires a (to the best of my knowledge) new regularity result for elliptic equations in a bounded, smooth domain that loose ellipticity on the boundary.

PABLO PADILLA, UNAM

A dynamical systems approach to symmetry in PDE's

We present a dynamical systems framework to obtain symmetry properties of partial differential equations with variational structure based on energy estimates. Comparisons with other methods, *e.g.* moving planes, symmetrization techniques, *etc.*, are also discussed.

PANAYOTIS PANAYOTAROS, IIMAS–UNAM

Localized invariant tori in the discrete NLS with diffraction management

We present results on the existence of localized invariant tori solutions in a discrete NLS equation with periodic parametric forcing. The equation models a system of coupled waveguide arrays with a special geometry that reduces diffraction effects. The solutions are obtained by continuing breather periodic solutions of an approximate autonomous system. We review some results on localized and multipeak solutions of this system and sketch a continuation argument that is based on general ideas on the continuation of invariant tori in Hamiltonian systems with symmetries.

MARIANITO ROCHA, Instituto Tecnológico Autónomo de México, Mexico City, México
Approximate monotonic traveling wave solutions to reaction-diffusion systems with general nonlinearities

We propose a simple variational approach to obtain approximate analytical expressions for monotonic traveling wave solutions of coupled reaction-diffusion systems with general nonlinearities.

Joint work with Robert Miura, New Jersey Institute of Technology.

HECTOR SANCHEZ MORGADO, Universidad Nacional Autónoma de México
Hyperbolicity and exponential convergence of the Lax Oleinik semigroup

Consider a convex superlinear Lagrangian $L\colon TM \to \mathbb{R}$ on a compact manifold M. It has been shown that there is a unique number c such that the Lax Oleinik semigroup $\mathcal{L}_t\colon C(M,\mathbb{R}) \to C(M,\mathbb{R})$ defined by

$$\mathcal{L}_t v(x) = \inf\left\{ v\big(\gamma(0)\big) + \int_0^t L(\gamma,\dot\gamma) + ct : \gamma\colon [0,t] \to M \text{ is piecewise } C^1, \gamma(t) = x \right\}$$

has a fixed point. Moreover for any $v \in C(M,\mathbb{R})$ the uniform limit $\tilde v = \lim_{t\to\infty} \mathcal{L}_t v$ exists.

Theorem 1 *Assume that the Aubry set consists in a finite number of hyperbolic periodic orbits or critical points of the Euler–Lagrange flow. Then, there is $\mu > 0$ such that for any $v \in C(M,\mathbb{R})$ there is $K > 0$ such that*

$$\|\mathcal{L}_t v - \tilde v\|_u \le K e^{-\mu t} \quad \forall t \ge 0.$$

We believe the reciprocal holds but for the moment we only have the proof for a mechanical Lagrangian.

Theorem 2 *Let $L\colon TM \to \mathbb{R}$ given by $L(x,v) = \frac{1}{2}v^2 - V(x)$ with*

$$\max_x V(x) = c, \quad V^{-1}(c) = \{x_1, \dots, x_m\}.$$

Suppose that there is $\mu > 0$ such that for any $v \in C(M,\mathbb{R})$ there is $K > 0$ such that

$$\|\mathcal{L}_t v - \tilde v\|_u \le K e^{-\mu t} \quad \forall t \ge 0.$$

Then $(x_i, 0)$, $i = 1, \dots m$ is a hyperbolic critical point of the Euler–Lagrange flow.