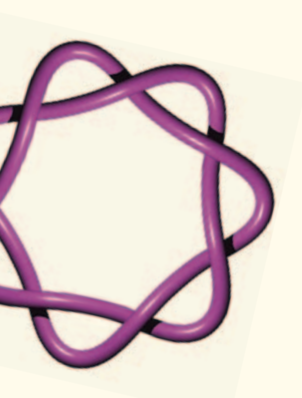
Notes from the Margin

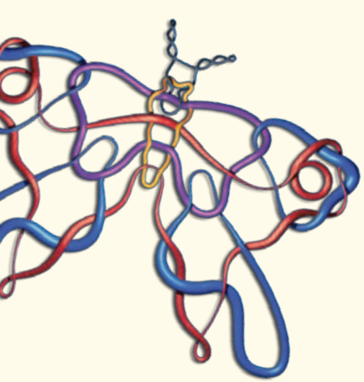
Volume I · 2011

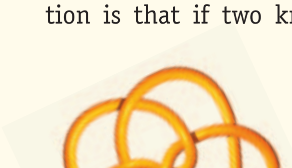
Foxxy Knots

by Garret Flowers

A knot is an intruiguing mathematical creature. On an intuitive level, we are all familiar with knots, which make their appearances in art, history and shoes. However, the formal theory of knots is surprisingly complex and lives in harmony between combinatorics, algebra, topology and geometry.







In this field, the questions are often simple to state, but difficult to answer. For instance, given two knots, how can we tell that they are equivalent? That is, can we pull, entangle, and manipulate the strands of one knot so that it becomes a copy of the second (without breaking the strands)? Indeed, how can we tell that a knot is 'knotted' at all and not just an entangled circle? While there has been quite a bit of progress on both of these questions, a simple and elegant solution remains elusive.

Take the knot diagrams of the trefoil and the 'unknot' pictured on the right in *Figure 1*. It is apparent that these two representations of knots are not equivalent. But how do we *prove* this? For this, we turn to knot invariants. A knot invariant assigns to each knot an object — be it a number, group, matrix, fruit or animal. The only condition is that if two knots are equivalent, then both knots must be assigned the same value. As 3D rendering software was not available to mathematicians in the mid-20th century, knots are often represented by a knot diagram. All of the knots in this article are represented by knot

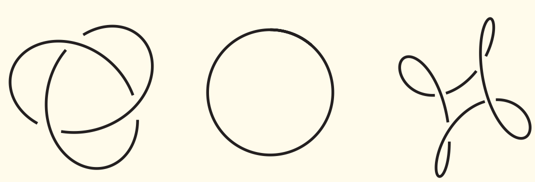


Figure 1. The two (un)knots on the right are equivalent; however, the trefoil knot on the left is not equivalent to the unknot.

diagrams — a curve on the plane that is broken in places where one strand of the knot moves beneath the other. Study of the knot diagrams is equivalent to the study of the knots themselves, although the former has a much nicer combinatorial description. In 1926, German

Photos of knots courtesy of Knotilus (knotilus.math.uwo.ca)

Preamble

by Kseniya Garaschuk

Kseniya Garaschuk, originally from Belarus, is a PhD candidate at University of Victoria. Her main research interests lie in the field on algebraic combinatorics and graph decompositions in particular. Apart from math, she loves baking and compensates for it by being a dedicated runner. A passionate fan of Formula One Racing, she collect model cars.

Whether in dull pencil or in threatening red pen, from a colleague or from your supervisor, an encouraging 'nice!' or a simple counterexample to your main result - much of every mathematician's life is spent in the margins. In fact, some of the most baffling mathematical problems of the modern era have spawned from someone's boastful notes in the margin. Somehow, we all find it easy to entrust little pieces of our minds to that narrow text-free part of a page.

I would like to welcome you to the CMS Student Committee's first addition of Notes from the Margin. Inside, you will find everything from news and announcements of the mathematical community to articles introducing some interesting research-level mathematics, opinion pieces and math-related stories. In this issue, our feature article is Foxxy Knots by Garret Flowers, a piece that resulted from the author's award-winning poster presented at the CMS Winter meeting 2010. The problem described in Chris Duffy's article was widely discussed at that same conference, albeit its formulation had to be tweaked to be suitable for publication. Finally, Jody Reimer's article spotlights a mathematician with an innovative and visually captivating approach to our subject. This is just the start: in the future, we hope to expand to include more pieces and a Distractions Page.

I really hope this magazine will become a place to share different tidbits of math. We are very audience driven, so I urge you to e-mail us at ***chair-studc@cms.math.ca*** with your feedback, comments and suggestions. And if you happen to find a truly marvelous mathematical morsel that just doesn't fit into the margin, do write it up on a separate piece of paper and submit it to us.

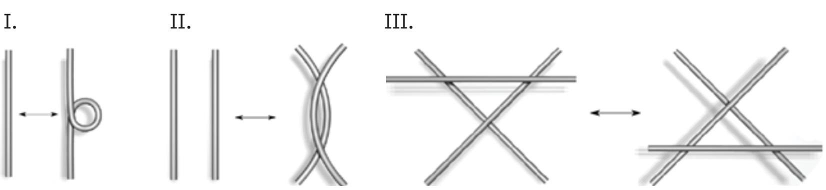


Figure 2. The three Reidemeister Moves (image courtacy of Yamashita, Makoto via Wikipedia)

mathematican Kurt Reidemeister developed what are now known as the Reidemeister moves: three transformations of knot diagrams that preserve the structure of the knot. The three moves are listed in *Figure 2*. It is clear that if we apply any of these three moves to a knot diagram, then the resulting knot is equivalent to the original. A powerful theorem of Reidemeister [R] states that these are the *only* three moves necessary to connect two equivalent knots. That is, two knots are equivalent if and only if a series of Reidemeister moves can be applied to one diagram to obtain a copy of the other diagram. The problem is in determining how many moves are needed, where they should be applied, and in what order.

But this issue hardly diminishes the power of the theorem. From this theorem, it becomes possible to construct a multitude of knot invariants. Think of the invariant as a function from the space of all knot diagrams to a collection of objects, for example, the integers. Then the invariant must assign to equivalent diagrams the same integer. Since two equivalent diagrams can always be connected by these three Reidemeister moves, it becomes sufficient to show that the invariant is unaffected by applying any of these three moves to a knot diagram. One of the older knot invariants developed in this manner is that of Fox tricolorations, named after the American mathematician Ralph Fox in the 1960s. A *tricoloration* of a knot diagram assigns to each segment of the diagram one of three colors, with just one rule: at each crossing, the three incident strands must all share the same color or must all be of distinct colors. We let $F_3(K)$ denote the number of ways to tricolor a knot diagram K.

For example, there are only three ways to tri-color the unknot, but there are nine distinct ways to tri-color a trefoil knot. Using the Reidmeister moves, it is not terribly difficult to show formally that this value is an invariant; however, some experimentation may be enough to convince yourself of its invariance. Since F_3 is an invariant, this proves that the trefoil and the unknot are actually distinct knots, as if they were the same, then they should have the same number of tricolorations. Since the trefoil has nine different colorations, and the unknot has only the

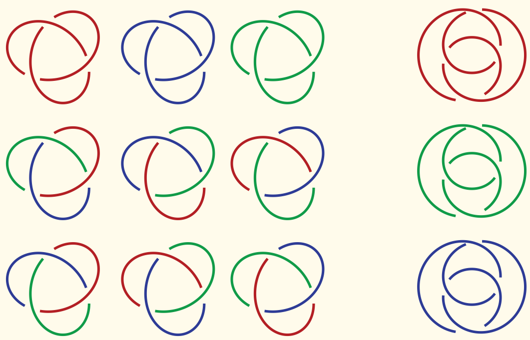


Table 1. All possible tricolorations of the trefoil and figure-eight knots

three trivial colorations, it follows that the two knots cannot be equivalent. Unfortunately, this invariant is not strong enough to distinguish between the unknot and the 'figure-eight' knot shown in *Table 1* (which is often referred to as the 4_1 knot).

In *Table 2*, a list of tricolorations for some simple knots is given. One particular property of F_3 is obvious: the tricoloration numbers are all powers of three. There are other, more subtle properties as well. We can deduce the number of tricolorations on some more complicated knots by

K	$F_3(K)$	$C_3(K)$
unknot	3	$\mathbb{Z}_3$
trefoil (3_1)	9	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
figure-eight (4_1)	3	$\mathbb{Z}_3$
trefoil#figure-eight ($3_1\#4_1$)	9	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
trefoil#trefoil ($3_1\#3_1$)	27	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

Table 2. Some small examples of $F_3(K)$ and $C_3(K)$

computing the tricolorations of simpler knots. For instance, we may apply a type of surgery to knots, known as connected sum composition. If K_1 and K_2 are two knot diagrams, then we can remove a small segment on each knot and then join the loose ends of K_1 to the ends of K_2, as depicted in *Figure 3*, with the trefoil and the figure-eight. This operation is frequent-

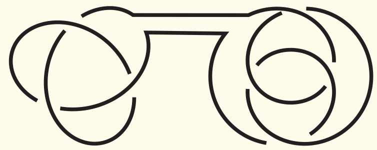


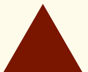
Figure 3. The connected sum $3_1\#4_1$ of the trefoil 3_1 and the figure-eight 4_1. It has 9 possible colorations.

ly denoted by $K_1\#K_2$. It is commutative, and it does not matter where we remove the segments of either knot. Through simple experimentation, or even a short formal proof, one can observe that the number of tricolorations on $K_1\#K_2$ can be deduced from $F_3(K_1)$ and $F_3(K_2)$ alone [P]. In fact, we have

$$F_3(K_1\#K_2) = \frac{1}{3}\left(F_3(K_1) \cdot F_3(K_2)\right).$$

Moreover, the set of tricolorations themselves forms a group structure on each diagram. Here, we assign each of the three colors a number in $\mathbb{Z}_3$. Then we 'add' two colorations by adding the colors strand-wise: the sum of the colors of a strand in the summands denotes the color of the same strand in the resulting knot (modulo 3). These groups, denoted by $C_3(K)$ are also indicated in *Table 2*. The first $\mathbb{Z}_3$ summand represents the three trivial colorings of the diagram.

While tricolorability is a nifty and unexpected invariant of knots, the colorings themselves are not particularly interesting. After all, our palatte of colors is extremely limited in scope. In order to make our knots truly colorful, we hope to enlarge our paint selection to n-colors, rather than simply three. This extension does not work in the naive sense: we lose our invariance. However, by modifying our rule somewhat, we are able to successfully color knots with a veritable rainbow of colors. First, take n colors, and represent each of the colors as an unique element of the group $\mathbb{Z}_n$. Then a Fox n-coloring of a diagram K assigns to each strand of the knot diagram a color in such a manner so that at each crossing the two under-strand colors sum to



Garret Flowers is currently attending graduate school at the University of Victoria. Mathematically, his interests lie somewhere in the space between combinatorics and geometry. When not knotting, Garret is an avid circus-artist, musician, and bubble-blower (see photo).

twice the color of the over-strand. This definition degenerates into our definition of a tricoloring when n=3. The number of Fox n-colorings of a diagram K is denoted by $F_n(K)$ and is also a knot invariant. The set of Fox n-colorings still forms an abelian group defined in the same manner as before, and we denote this group as $C_n(K)$.

Figure 4. Two nontrivial 5-coloring of the figure-eight (4_1) knot, where the numbers 0, 1, 2, 3, 4 in $\mathbb{Z}_5$ are associated to red, green, blue, yellow and purple, respectively.

Using these invariants, we can now distinguish between the unknot and the 4_1 knot. It is always the case that F_n(unknot)=n and C_n(unknot)=$\mathbb{Z}_n$; however, we find that $F_5(4_1)>n$, since the two colorings in *Figure 4* are examples of nontrivial 5-colorings of the 4_1 knot. In fact, there are 25 possible 5-colorings of the figure-eight.

The properties mentioned above for tricolorings extend naturally to n-colorings. That is,

$$F_n(K)=n^m, \text{ for some } m \in \mathbb{N}$$

$$F_n(K_1\#K_2)=\frac{1}{n}(F_n(K_1)\cdot F_n(K_2))$$

There are many other properties relating various n-colorings as well, some are easily discernable through combinatorial arguments, while others are more interesting and suggest some deeper mathematical properties of knots. For instance, there exist surprising connections between the

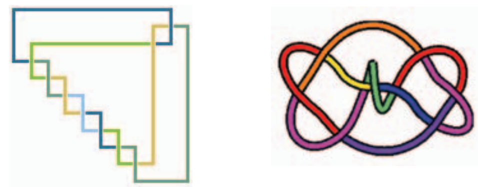


Figure 5. A 5-coloring of the 9_2 knot, and an 11-coloring of the 9_{10} knot

group $C_n(K)$ and the fundamental group of the compliment of the knot. But strengthening the link between mathematical knots and artistic knots is reason enough to play with these diagrams.

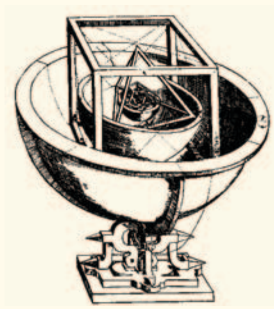
[R] Kurt Reidemeister, Elementare Begründung der Knotentheorie, Abh. Math. Sem. Univ. Hamburg 5 (1926), 24-32

[P] Józef H. Przytycki. 3-coloring and other elementary invariants of knots. ArXiv

Da Vinci in 3D... Without the Goggles!

Jody R. Reimer – University of Manitoba

If there is one surefire way to get Dr. David Gunderson of the University of Manitoba Department of Mathematics excited, it is by asking him to show you what he is currently working on in his garage.



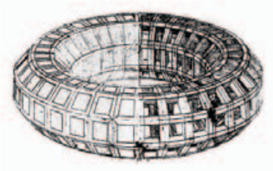
Following in the footsteps of ancient Greeks, such as Archimedes and Plato, and armed with a (very expensive!) copy of Luca Pacioli's *De Divina Propotione*, his passion for geometry has spilled over to combine with his other love – woodworking. Like many who have gone before him, Dr. Gunderson is fascinated by the predictability, precision, and aesthetic charm of this division of mathematics and his creations illustrate clearly its beauty and complexity.

In the late 1980's, Gunderson took a summer job at Artek Manufacturing, a fine woodworking factory. It was there that he acquired the attention to detail and the necessary finesse to create the models which would later follow. A decade later, Gunderson saw the famous triangle-with-a-hole puzzle and subsequently attempted to produce it to show to his classroom using a hand saw. The result clearly was not a piece of fine art, so he purchased a compound miter saw.

Then he wondered if he could master a dodecahedron. The rest is, as they say, history.

Much of his inspiration comes from *De Divina Propotione*. This work contains 60 drawings by Leonardo da Vinci of various solids and see-through frameworks, several of which Gunderson has constructed. Many of these sculptures are polyhedra (3D solids with polygonal faces), and each fit into further subcategories. Only five regular polyhedra exist (with all faces being the same regular polygon and with the same number meeting at each vertex), while the class of semi-regular polyhedra is larger (examples being the Archimedean solids, regular prisms, and antiprisms – two parallel polygons connected by a band of alternating triangles). *The Campanus Sphere* is a special rarity, found in very few other collections of the sort and, with five-inch lengths, Gunderson's is one of the largest, valued at $2500! Further models have resulted from the works of Johannes Kepler, who, while best known for his laws of planetary motion, also extensively studied polyhedra: for example, it was Kepler who discovered the infinite class of antiprisms. In addition to these, Gunderson has also crafted several other wooden models, including puzzles, a Möbius band, trefoil knots, and space-filling polyhedra.

Determining cutting angles and part placement is a task not to be done through trial and error, especially for forms constructed as see-through frameworks.



Made from a wide array of woods (approximately 35 different varieties), and with each shape posing unique obstacles, the logistics of bringing da Vinci's sketches to life present some challenges. Determining cutting angles and part placement is a task not to be done through trial and error, especially for forms constructed as see-through frameworks. *The Campanus Sphere* previously mentioned required three pages of meticulous calculations. In addition to mathematical difficulties, there are various issues of technique, with everything going wrong from stain that refuses to dry, to models ripped out of hand by belt sanders, and an incident involving the loss of fingertips (details omitted).

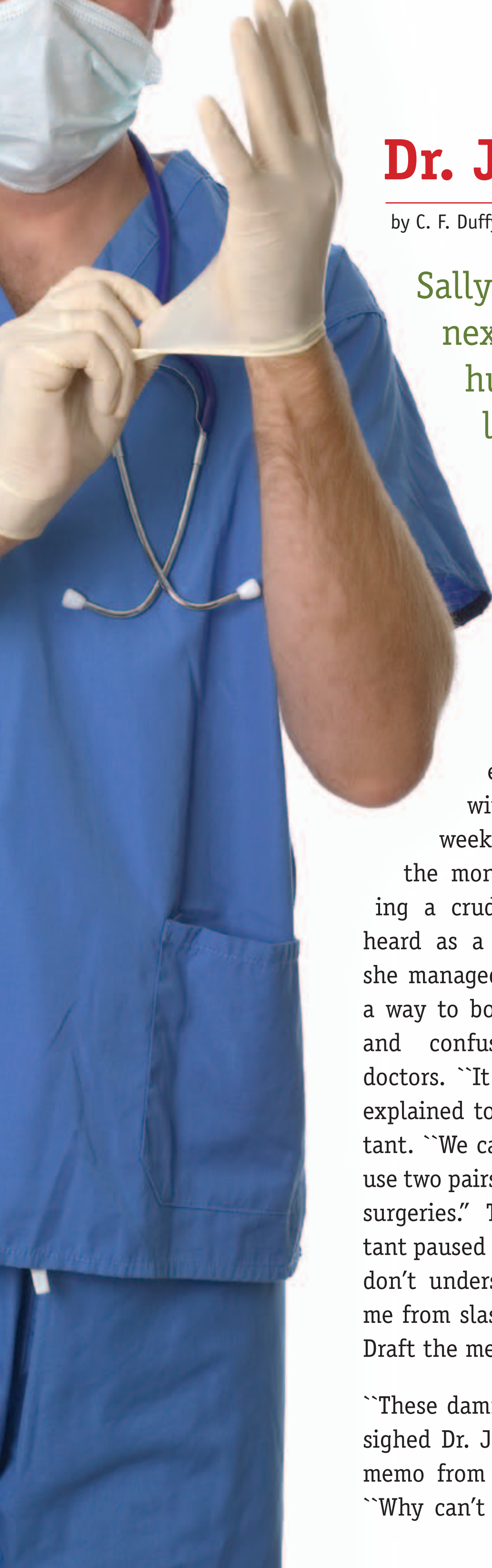
Gunderson's current project is the *Mysterium Cosmographicum*, which, when completed, will be the first of its kind. Dreamed up by Johannes Kepler, it is a model proposing the relationship between the six planets known at the time (unfortunately, this did not turn out to be accurate, but was nonetheless a nice idea). He believed, and his 1596 book by the same name posited it, the model comprised of the five Platonic solids layered between the six planetary spheres. Allegedly, his plan was to have it functioning as a punchbowl dispensing assorted beverages, but it was never completed. Perhaps Gunderson will one day have the honour.

Thus, what started as a means to demonstrate mathematical concepts to students and a unique gift idea for friends and colleagues has now overflowed Dr. Gunderson's office and spilled into the main hall of the mathematics building. With whatever new projects come up in the future, these models will continue to demonstrate delicate craftsmanship as well as the beauty and diversity of geometry.

The images are courtesy of Virtual Polyhedra project and Dr. Gunderson. See their web-sites for more:

http://www.georgehart.com/virtual-polyhedra/kepler.html
http://home.cc.umanitoba.ca/~gunderso/

Jody Reimer Currently in the final year of her undergraduate degree, Jody has recently decided to shift her focus towards applied math, specifically the relationship between math and ecology. In the time remaining after homework and work as a teaching assistant, she enjoy baking bread, extended coffee breaks with friends, and being outside. Jody is eager to continue on to master's studies next year at Oxford University.



Dr. Johnson’s latex problem

by C. F. Duffy

Sally had big hopes for her career. She was going to be the next John Forbes Nash, but an unfortunate liaison with the husband of the head of the department of mathematics had left her career in tatters. ``Sure”, she reasoned, ``I seduced him, but he was the one who let the secret slip.”

Now, after having her good name tarnished, she couldn’t find a post-doc position and she spent her days working in the finance department of the local hospital. Most days her mental talents were wasted; being a number-bot isn’t especially hard work for a woman with an IQ north of 150. However, last week was a change from the monotony. Remembering a crude riddle she first heard as a graduate student, she managed to come up with a way to both cut the budget and confuse simple-minded doctors. ``It is so simple”, she explained to the chief accountant. ``We can have one doctor use two pairs of gloves for three surgeries.” The chief accountant paused for a moment — ``I don’t understand, but that has never stopped me from slashing the budget before. Let’s do it. Draft the memo”.

``It is so simple”, she explained to the chief accountant. ``We can have one doctor use two pairs of gloves for three surgeries.”

``These damned budget cuts are getting worse”, sighed Dr. Johnson, as he picked up the latest memo from the hospital’s finance department. ``Why can’t we just go to an American model? Sure, everyone else is screwed, but at least I won’t have to have placeboes prescribed to my wife instead of her anti-psychotic medication”, he wondered aloud. He thought that after the hospital replaced all of the tape and gauze with Spiderman Band-Aids, it couldn’t get any worse. Indeed, it had. The most recent round of cutbacks had been the worst yet. It wasn’t enough to start having his orthopaedic department share supplies with those frauds in psych ward, but now the accountants were telling him to cut back on his department’s use of rubber gloves. ``This doesn’t even make any sense. How can I use two pairs of gloves to complete three surgeries without putting anyone at risk? I certainly don’t want their blood on my hands no more than they want another’s blood inside them. They’ve gone too far.” With that he promptly opened his computer, sent a sharply worded missive off to the CFO and turned to a more pressing problem: finding ice for his quickly warming glass of scotch.

Sally opened her eyes from her daily midday nap at her desk. Wiping the sleep from her eyes, she noticed the flashing light on her

The student poster sessions have been gaining momentum and are receiving increased attention from the community.

Participate in the next one that will be held during the CMS Summer 2011 meeting in Edmonton. Present your research in a relaxed one-on-one atmosphere and compete for prizes! Check the conference web-site for tips that will help you create a winning poster and e-mail Kseniya at ***chair-studc@cms.math.ca*** if you have any questions.

Blackberry handset. It was probably another of her graduate school friends telling her about their latest research results in something obscure like Kloosterman sums, she thought. It wasn't.``Dr. Johnson? Who the hell is Dr. Johnson and why has his email been forwarded to me. He's a doctor, he should be smart enough to figure it out", she mumbled, still half asleep. ``I suppose I could give him a hint", she sighed.

As he reached into his office mini-fridge, he heard the familiar chime that told him he had another new email. ``Probably just another request for a consult", he snorted with derision. ``When will people learn not to combine alcohol and skiing?". Returning to his desk without the ice (he had forgot to have his assistant refill the tray after his morning scotch), he opened his email. It was not a message from a family doctor, but a return email from someone in the accounting department named Sally. It was as short as it was cryptic. ``Wear both pairs". To Dr. Johnson this made even less sense, and he let the accountant know in his expletive-ridden reply.

Sally had just managed to drift off to her 2 p.m. nap when she was jarred awake by the sharp chime of her Blackberry handset. She quickly realized that this doctor wasn't about to figure this out on his own. ``I don't give a damn who this Dr. Johnson is. He isn't going to intimidate me", she thought. Much in the way she used to handle her supervisor, she figured the best way to deal him was in person, and set off down the hall to have a little chat with Dr. Johnson.

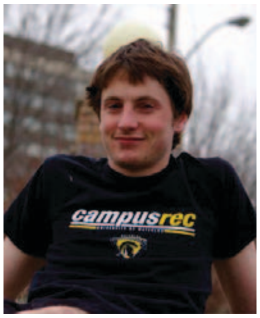
Much in the way she used to handle her supervisor, she figured the best way to deal him was in person, and set off down the hall to have a little chat with Dr. Johnson.

Dr. Johnson sipped the last of his scotch and finished looking through his notes before his surgery-filled afternoon. He most recent reply to the accounting department had gone unanswered. ``That made my point", he proudly chuckled to himself as began to mentally prepare himself for the challenges ahead. Suddenly, his concentration was broken by a pair of sharp knocks on his door. ``Come in!" he bellowed to this clearly unwanted visitor.

It was over as quickly as it had begun. As Sally waltzed from his office, Dr. Johnson looked on in disbelief. ``It was so simple, Richard. Why didn't I think of it myself?" he sheepishly muttered to himself. He leaned back in his chair and closed his eyes. As he drifted off to sleep he began to wonder what other applications he could find for this newly-learned technique. Though he didn't realize it then, this technique would be his saviour when he next encountered Sally a few months later at the hospital's Christmas party...

What did Sally say to Dr. Johnson?

Put on both pairs of gloves and perform the first surgery. Remove the outer pair and perform the second surgery. Turn the outer pair inside-out and put them back over the inner pair. Perform the third surgery. Given n pairs of gloves, what is the maximum number of successful surgeries that Dr. Johnson can perform?



Christopher F. Duffy is a MSc student at the University of Victoria. He was coerced into contributing to this project by his officemate, the editor.

Where to look to find a job?

by Sarah Plosker

Sarah Plosker is just getting started on a PhD in Applied Mathematics at the University of Guelph. Her research interests lie within the fields of Operator Algebras and Quantum Information Theory. She hopes to one day make significant contributions to the area and have a major theorem named after her.

The Student Committee held a student panel discussion on the Hiring Process at the Vancouver 2010 Winter CMS Meeting. Here are some links our panelists have suggested for students currently on the market for a job.

American Mathematical Society:
eims.ams.org/search.cfm

Canadian Mathematical Society:
cms.math.ca/

MathJobs:
www.mathjobs.org

Mathematics Jobs Wiki:
notable.math.ucdavis.edu/wiki/Mathematics_Jobs_Wiki

European Mathematical Society:
www.euro-math-soc.eu/

Institute for Mathematics and its Applications: www.ima.umn.edu/

London Mathematical Society:
www.lms.ac.uk/ and www.jobs.ac.uk/

Times Higher Education:
www.timeshighereducation.co.uk/

Five minute interviews at MAA AMS joint meetings:
www.maa.org/meetings/jmm.html

These are great resources. Even if you are not job hunting at this point, take a look to see what the market is like today and what other info is available on these sites.

Other Contributors:

David Thomson is a PhD candidate at Carleton University. His main research interests are on all algebraic and number theoretic aspects of finite fields and their applications. Currently David is enjoying a research term in Dublin, Ireland at the Claude Shannon Institute. In his spare time he plays every sport except hockey, fumbles around on his acoustic guitar and enjoys a nice bottle of Italian red whilst cooking amazing risotto.

Would you like to submit an article to *Notes from the Margin*?

Tell us about a math problem you are interested in, a mathematician that inspires you, write an opinion piece, create a puzzle or a crossword.

Send you submissions to ***student-editor@cms.math.ca.***

CMS Summer Meeting

The 2011 CMS Summer Meeting will take place June 3 - 5, 2011, to be hosted by the University of Alberta in Edmonton.

The CMS Meetings provide students with many opportunities, such as networking, learning, presenting and simply having fun! For more information about scientific sessions, go to the meeting website at ***http://cms.math.ca/Events/summer11/***

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