PROBLEMS FOR AUGUST

Please send your solution to
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no later than September 30, 2003. It is important that your complete mailing address and your email address appear on the front page.

227. [*Since the original statement of this problem in May was incorrect and not everyone picked up the correction, it is reposed.*] Let n be an integer exceeding 2 and let $a_0, a_1, a_2, \cdots, a_n, a_{n+1}$ be positive real numbers for which $a_0 = a_n$, $a_1 = a_{n+1}$ and

$$a_{i-1} + a_{i+1} = k_i a_i$$

for some positive integers k_i, where $1 \leq i \leq n$.

Prove that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n \ .$$

241. [*Corrected.*] Determine

$$\sec 40^\circ + \sec 80^\circ + \sec 160^\circ \ .$$

248. Find all real solutions to the equation

$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1 \ .$$

249. The non-isosceles right triangle ABC has $\angle CAB = 90^\circ$. Its inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle of triangle ABC meets UV in S. Prove that:

(a) $ST \| BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle, and d_1 and d_2 are the respective distances from S to AC and AB.

250. In a convex polygon P, some diagonals have been drawn so that no two have an intersection in the interior of P. Show that there exists at least two vertices of P, neither of which is an enpoint of any of these diagonals.

251. Prove that there are infinitely many positive integers n for which the numbers $\{1, 2, 3, \cdots, 3n\}$ can be arranged in a rectangular array with three rows and n columns for which (a) each row has the same sum, a multiple of 6, and (b) each column has the same sum, a multiple of 6.

252. Suppose that a and b are the roots of the quadratic $x^2 + px + 1$ and that c and d are the roots of the quadratic $x^2 + qx + 1$. Determine $(a - c)(b - c)(a + d)(b + d)$ as a function of p and q.

253. Let n be a positive integer and let $\theta = \pi/(2n + 1)$. Prove that $\cot^2 \theta$, $\cot^2 2\theta$, $\cdots$, $\cot^2 n\theta$ are the solutions of the equation

$$\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \binom{2n+1}{5} x^{n-2} - \cdots = 0 \ .$$

254. Determine the set of all triples (x, y, z) of integers with $1 \leq x, y, z \leq 1000$ for which $x^2 + y^2 + z^2$ is a multiple of xyz.