2017 CMO
Qualifying Repêchage

Sun
Life Financial

Official Problem Set

1. Malcolm writes a positive integer on a piece of paper. Malcolm doubles this integer and subtracts 1, writing this second result on the same piece of paper. Malcolm then doubles the second integer and adds 1, writing this third integer on the paper. If all of the numbers Malcolm writes down are prime, determine all possible values for the first integer.

2. For any positive integer n, let $\phi(n)$ be the number of integers in the set $\{1, 2, \ldots, n\}$ whose greatest common divisor with n is 1. Determine the maximum value of $\dfrac{n}{\phi(n)}$ for n in the set $\{2, \ldots, 1000\}$ and all values of n for which this maximum is attainted.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following equation for all $x, y \in \mathbb{R}$.

$$(x + y)f(x - y) = f(x^2 - y^2).$$

4. In this question we re-define the operations addition and multiplication as follows: $a + b$ is defined as the minimum of a and b, while $a*b$ is defined to be the sum of a and b. For example, $3+4 = 3$, $3*4 = 7$, and $3*4^2+5*4+7 = \min(3 \text{ plus } 4 \text{ plus } 4, 5 \text{ plus } 4, 7) = \min(11, 9, 7) = 7$.

   Let a, b, c be real numbers. Characterize, in terms of a, b, c, what the graph of $y = ax^2+bx+c$ looks like.

5. Prove that for all real numbers x, y,

$$(x^2 + 1)(y^2 + 1) + 4(x - 1)(y - 1) \geq 0.$$

   Determine when equality holds.

6. Let N be a positive integer. There are N tasks, numbered $1, 2, 3, \ldots, N$, to be completed. Each task takes one minute to complete and the tasks must be completed subjected to the following conditions:

   • Any number of tasks can be performed at the same time.

   • For any positive integer k, task k begins immediately after all tasks whose numbers are divisors of k, not including k itself, are completed.

   • Task 1 is the first task to begin, and it begins by itself.

   Suppose $N = 2017$. How many minutes does it take for all of the tasks to complete? Which tasks are the last ones to complete?

7. Given a set $S_n = \{1, 2, 3, \ldots, n\}$, we define a *preference list* to be an ordered subset of S_n. Let P_n be the number of preference lists of S_n. Show that for positive integers $n > m$, $P_n - P_m$ is divisible by $n - m$.

   Note: the empty set and S_n are subsets of S_n.

8. A convex quadrilateral $ABCD$ is said to be *dividable* if for every internal point P, the area of ΔPAB plus the area of ΔPCD is equal to the area of ΔPBC plus the area of ΔPDA. Characterize all quadrilaterals which are dividable.