Solutions to December problems.

416. Let P be a point in the plane.

(a) Prove that there are three points A, B, C for which $AB = BC$, $\angle ABC = 90^\circ$, $|PA| = 1$, $|PB| = 2$ and $|PC| = 3$.

(b) Determine $|AB|$ for the configuration in (a).

(c) A rotation of 90° about B takes C to A and P to Q. Determine $\angle APQ$.

Solution 1. (a) We first show that a figure similar to the desired figure is possible and then get the lengths correct by a dilatation. Place a triangle in the cartesian plane with A at $(0,1)$, B at $(0,0)$ and C at $(1,0)$. Let P be at (x,y). The condition that $PA : PB = 1 : 2$ yields that

$$x^2 + y^2 = 4[x^2 + (y-1)^2] \Longleftrightarrow 0 = 3x^2 + 3y^2 - 8y + 4 \ .$$

The condition that $PB : PC = 2 : 3$ yields that

$$9[x^2 + y^2] = 4[(x-1)^2 + y^2] \Longleftrightarrow 0 = 5x^2 + 5y^2 + 8x - 4 \ .$$

Hence $3x + 5y = 4$, so that $9x^2 = 16 - 40y + 25y^2$ and

$$0 = 2(17y^2 - 32y + 14) \ .$$

Solving these equations yields

$$(x,y) = \left(\frac{5\sqrt{2} - 4}{17}, \frac{16 - 3\sqrt{2}}{17} \right) ,$$

a point that lies within the positive quadrant, and

$$(x,y) = \left(\frac{-5\sqrt{2} - 4}{17}, \frac{16 + 3\sqrt{2}}{17} \right) ,$$

a point that lies within the second quadrant..

(b) In the first situation,

$$|PB|^2 = \frac{20 - 8\sqrt{2}}{17} \ .$$

Rescaling the figure so that $|PB| = 2$, we find that the rescaled square has side length equal to the **square root** of

$$(17)/(5 - 2\sqrt{2}) = 5 + 2\sqrt{2} \ .$$

In the second situation,

$$|PB|^2 = \frac{20 + 8\sqrt{2}}{17} \ .$$

Rescaling the figure so that $|PB| = 2$, we find that the rescaled square has side length equal to the **square root** of

$$(17)/(5 + 2\sqrt{2}) = 5 - 2\sqrt{2} \ .$$

(c) Since triangle BPQ is right isosceles, $|PQ| = 2\sqrt{2}$. Since also $|AQ| = |CP| = 3$ and $|AP| = 1$, $\angle APQ = 90^\circ$, by the converse of Pythagoras' theorem.

Comment. *Ad* (a), P is on the intersection of two Apollonius circles with diameter joining $(-2,0)$ and $(2/5,0)$ passing through the points $(0, 2/\sqrt{5})$ on the $y-$axis and with diameter joining $(0,2)$ and $(0,2/3)$. These intersect within the triangle and outside of the triangle.

Solution 2. Suppose that the square has side length x. Let the perpendiculat distance from P to AB be a and from P to BC be b, both distances measured within the right angle. Then we have the three equations: (1) $a^2 + b^2 = 4$; (2) $a^2 + (x - b)^2 = 1$ or $x^2 = 2bx - 3$; (3) $b^2 + (x - a)^2 = 9$ or $x^2 = 2ax + 5$. Hence $2x(b - a) = 8$, so that $x = 4(b - a)^{-1}$. Also $4x = x^2(b - a) = 5b + 3a$, which along with $b - a = 4/x$ yields

$$2a = x - \frac{5}{x} \qquad \text{and} \qquad 2b = \frac{3}{x} + x \ .$$

Thus

$$\begin{aligned} 16 &= \left(x - \frac{5}{x}\right)^2 + \left(\frac{3}{x} + x\right)^2 = 2x^2 + \frac{34}{x^2} - 4 \\ &\Longrightarrow x^4 - 10x^2 + 17 = 0 \\ &\Longrightarrow x^2 = 5 \pm 2\sqrt{2} \ . \end{aligned}$$

For $2a$ to be positive, we require that $x^2 > 5$ and so $x = \sqrt{5 + 2\sqrt{2}}$ and P is inside triangle ABC. Since

$$(5 - 2\sqrt{2})^2 < \left(5 - \frac{2 \times 7}{5}\right)^2 = \left(\frac{11}{5}\right)^2 < 5 \ ,$$

the second value of x yields negative a and the point lies on the opposite side of AB to C.

For (c), we consider two cases:

(1) $|AB| = \sqrt{5 + 2\sqrt{2}}$ and P lies inside the triangle ABC. Applying the law of cosines to triangle APB yields $\cos \angle APB = -1/\sqrt{2}$ and $\angle APB = 135°$. Hence $\angle APQ = \angle APB - \angle QPB = 135° - 45° = 90°$.

(2) $|AB| = \sqrt{5 - 2\sqrt{2}}$ and P lies outside the triangle ABC. Then the law of cosines applied to triangle APB yields $\cos \angle APB = 1/\sqrt{2}$ and $\angle APB = 45°$. Hence $\angle APQ = \angle APB + \angle BPQ = 45° + 45° = 90°$.

Solution 3. [D. Dziabenko] We can juxtapose two right triangles of sides $(2, 2, 2\sqrt{2})$ and $(1, 2\sqrt{2}, 3)$ to obtain a quadrilateral with $|XY| = |XW| = 2$, $|YZ| = 1$, $|ZW| = 3$ and $|YW| = 2\sqrt{2}$. Since $\angle XYZ = 135°$, we can use the law of cosines to find that $|XZ| = \sqrt{5 + 2\sqrt{2}}$.

A rotation of $90°$ about X takes W to Y and Z to T, so that $|YZ| = 1$, $|XY| = 2$, $|YT| = |WZ| = 3$ and $|XZ| = |XT| = \sqrt{5 + 2\sqrt{2}}$. Relabel Y as P, X as B, Z as A and T as C to get the desired configuration. For (b), we have that $|AB| = |XZ| = \sqrt{5 + 2\sqrt{2}}$, and, for (c), that $Q = W$ and $\angle APQ = \angle XYW = 90°$.

Solution 4. [J. Kileel] Let $P \sim (0, 0)$, $B \sim (0, 2)$, $A \sim (a, b)$, $C \sim (c, d)$. The conditions to be satisfied are: (1) $a^2 + b^2 = 1$; (2) $c^2 + d^2 = 9$; (3) $a^2 + (b - 2)^2 = c^2 + (d - 2)^2 \Longrightarrow d = b + 2$;

$$(4) \qquad \frac{b - 2}{a} = \frac{c}{2 - d} = \frac{c}{-b} \Longrightarrow -b^2 + 2b = ac \Longrightarrow b^4 - 4b^3 + 4b^2 = (1 - b^2)(5 - b^2 - 4b) \ .$$

Hence

$$0 = 8b^3 - 10b^2 - 4b + 5 = (4b - 5)(2b^2 - 1) \ .$$

Since $b = -5/4$ is extraneous (why?), either $b = 1/\sqrt{2}$ or $b = -1\sqrt{2}$.

Lat $b = 1/\sqrt{2}$. From symmetry, it suffices to take $a = 1/\sqrt{2}$ and we get

$$(a, b, c, d) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2\sqrt{2} - 1}{\sqrt{2}}, \frac{2\sqrt{2} + 1}{\sqrt{2}}\right) \ ,$$

and

$$|AB|^2 = |BC|^2 = 5 - 2\sqrt{2} \ .$$

Lat $b = -1/\sqrt{2}$. Again we take $a = 1/\sqrt{2}$ and we get

$$(a, b, c, d) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-2\sqrt{2}-1}{\sqrt{2}}, \frac{2\sqrt{2}-1}{\sqrt{2}}\right) ,$$

and

$$|AB|^2 = |BC|^2 = 5 + 2\sqrt{2} .$$

Thus the configuration is possible and we have the length of $|AB|$. In the first case, the rotation about B that takes C to A is clockwise and carries P to $Q \sim (-2, 2)$. It is straightforward to check that $\angle APQ = 90°$. In the second case, the rotation about B that takes C to A is counterclockwise and carries P to $Q \sim (2, 2)$. Again, $\angle APQ = 90°$.

Solution 5. Place B at $(0, 0)$, A at $(0, a)$, C at $(a, 0)$ and P at (b, c). Then we have to satisfy the three equations: (1) $(a - c)^2 + b^2 = 1$; (2)$b^2 + c^2 = 4$; (3) $(b - a)^2 + c^2 = 9$. Taking the differences of the first two and of the last two lead to the equations

$$2c = a + \frac{3}{a} \qquad 2b = a - \frac{5}{a}$$

from which, through substitution in (2), we get that $a^4 - 10a^2 + 17 = 0$. This leads to the possibilities that $a^2 = 5 \pm 2\sqrt{2}$, and we can complete the argument as in the foregoing solutions.

417. Show that for each positive integer n, at least one of the five numbers 17^n, 17^{n+1}, 17^{n+2}, 17^{n+3}, 17^{n+4} begins with 1 (at the left) when written to base 10.

Solution 1. It is equivalent to show that, for each natural number n, one of 1.7^{n+k} $(0 \leq k \leq 4)$ begins with the digit 1. We begin with this observation: if for some positive integers u and r, $1.7^u < 10^r \leq 1.7^{u+1}$, then

$$1.7^{u+1} = (1.7)(1.7)^u < (1.7)10^r < 2 \cdot 10^r$$

and the first digit of 1.7^{u+1} is 1.

We obtain the desired result by induction. $1.7^1 = 1.7$ begins with 1, so one of the first five powers of 1.7 begins with 1. Suppose that for some positive integer n exceeding 4, one, at least, of every five consecutive powers of 1.7 up to 1.7^n begins with 1. Let $m \leq n$ be the largest positive integer for which $10^v < 1.7^m < 2 \cdot 10^v$ for some integer v. Then $1.7^m < 10^{v+1}$ and

$$1.7^{m+5} = (1.7)^m (1.7)^5 = (1.7)^m (14.19857) > 10^{v+1}$$

with the result that, for u equal to one of the numbers $m, m+1, m+2, m+3, m+4$, $1.7^u < 10^{v+1} \leq 1.7^{u+1}$. Hence, one of the numbers 1.7^{m+k} $(1 \leq k \leq 5)$ begins with the digit 1. If it is 1.7^{m+k}, then $m + k > n$ and we have established the result up to $m + k$.

Solution 2. For $n = 1$, 17^n begins with 1. Suppose that, for some positive integer k, 17^k begins with 1. Then, either

$$10^a < 17^k < \frac{10^5}{17^4} 10^a$$

or

$$\frac{10^5}{17^4} 10^a < 17^k < 2 \times 10^a$$

for some positive integer a. In the former case,

$$10^{a+5} < 17^5 \times 10^a < 17^{k+5} < 17 \times 10^{a+5}$$

so that 17^{k+5} begins with 1. In the latter case,

$$10^{a+5} < 17^{k+4} < 2 \times 10^a \times 17^4 < 2 \times 10^a \times 300^2 = 1.8 \times 10^{a+5}$$

so that 17^{k+4} begins with 1. The result follows.

Solution 3. Let $17^n = a \cdot 10^m + b$ where $0 \le b < 10^m$. Then

$$a \times 10^m < 17^n < (a+1)10^m$$

so that

$$(1.7a)10^{m+1} < 17^{n+1} < (1.7)(a+1)10^{m+1} .$$

Let $6 \le a \le 9$. Then

$$10^{m+2} < (1.7)6 \times 10^{m+1} < 17^{n+1} < 1.7 \times 10^{m+2}$$

and 17^{n+1} begins with 1. Let $4 \le a \le 5$. Then

$$6 \times 10^{m+1} < 4(17)10^m \le (17a)10^m < 17^{n+1} < (1.7)6 \times 10^{m+1} < (1.02)10^{m+1}$$

so that, either 17^{n+1} begins with 1, or 17^{n+1} begins with 6, 7, 8 or 9 and 17^{n+2} begins with 1. When $a = 3$, $5 \times 10^{m+1} < 17^{n+1} < 7 \times 10^{m+1}$ and either 17^{n+2} or 17^{n+3} begins with 1. When $a = 2$, then $3 \times 10^{m+1} < 17^{n+1} < 6 \times 10^{m+1}$ and one of 17^{n+2}, 17^{n+3}, 17^{n+4} begins with 1. Finally, if $a = 1$, one can similarly show that one of 17^{n+k} $(1 \le k \le 5)$ begins with 1. The argument now can be completed by induction.

Solution 4. [D. Dziabenko] 17^n beginning with 1 is equivalent to $10^m < 17^n < 2 \times 10^m$ for some positive integer m, which in turn is equivalent to

$$m < n \log 17 < m + \log 2$$

or

$$p < n \log 1.7 < p + \log 2$$

for some positive integer $p (= m - n)$.

Suppose that 17^n begins with 1. We observe that $\log 1.7 < \log 2 = (1/3) \log 8 < 1/3$ and that $2^{10} > 10^3$, whereupon $\log 2 > 3/10$ and

$$\log 1.7 = (\log 17) - 1 > (\log 16) - 1 = (4 \log 2) - 1 > \frac{6}{5} - 1 = \frac{1}{5}$$

$$\Longrightarrow 1 < 5 \log 1.7 < 5 \log 2 < 5/3$$

and so the integer part of $(n+5)\log 17$ is exactly one more than the integer part of $n \log 17$.

From the foregoing, each interval of length $\log 2$ must contain a multiple of $\log 1.7$ and in particular the interval

$$\{x : p + 1 < x < p + 1 + \log 2\}$$

must contain at least one of $(n+k) \le 1.7$ $(1 \le k \le 5)$. We can now complete the argument for the result by induction.

418. (a) Show that, for each pair m,n of positive integers, the minimum of $m^{1/n}$ and $n^{1/m}$ does not exceed $3^{1/2}$.

(b) Show that, for each positive integer n,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 \ge n^{1/n} \ge 1 .$$

(c) Determine an integer N for which

$$n^{1/n} \leq 1.00002005$$

whenever $n \geq N$. Justify your answer.

Solution. (a) Wolog, we may assume that $m \leq n$, so that $m^{1/n} \leq n^{1/n}$. It suffices to show that, for each positive integer n, $n^{1/n} \leq 3^{1/3} (< 3^{1/2})$ or that $n \leq 3^{n/3}$. Since $3 > 64/27$, it follows that $3^{1/3} - 1 > (4/3) - 1 = 1/3 > 0$ and the result holds for $n = 1$. Suppose as an induction hypothesis, that it holds for n. Then, since $3^{n/3} \geq n$,

$$\begin{aligned} 3^{(n+1)/3} &\geq (3 + \overline{n-3})3^{1/3} > 3^{4/3} + n - 3 \\ &= n + 3(3^{1/3} - 1) > n + 1 \ . \end{aligned}$$

(b) Note that

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + n\left(\frac{1}{\sqrt{n}}\right) = 1 + \sqrt{n} > \sqrt{n} \ .$$

Alternatively, we can note that, by taking a term out of the binomial expansion,

$$\begin{aligned} (\sqrt{n} + 1)^{2n} &> \binom{2n}{2} (\sqrt{n})^{2n-2} = \frac{2n(2n-1)}{2} n^{n-1} \\ &= (2n-1)n^n \geq n^{n+1} \ , \end{aligned}$$

from which

$$\left(1 + \frac{1}{\sqrt{n}}\right)^{2n} = \frac{(\sqrt{n}+1)^{2n}}{n^n} > n \ .$$

(c) By (b), it suffices to make sure that $(1 + n^{-1/2})^2 \leq 1.00002005$. Let $N = 10^{10}$. Then, for $n \geq N$, we have that $\sqrt{n} \geq 10^5$, so that

$$(1 + n^{-1/2})^2 \leq (1.00001)^2 = 1.0000200001 < 1.00002005 \ .$$

419. Solve the system of equations

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$$

for x, y, z not all equal. Determine xyz.

Solution 1. Taking pairs of the three equations, we obtain that

$$x - y = \frac{y-z}{yz}, \qquad y - z = \frac{x-z}{xz}, \qquad z - x = \frac{x-y}{xy} \ .$$

Since equality of any two of x, y, z implies equality of all three, x, y, z must be distinct. Multiplying these three equations together we find that $(xyz)^2 = 1$.

When $xyz = 1$, then $z = 1/xy$ and we find that solutions are given by

$$(x, y, z) = \left(x, -\frac{1}{x+1}, -\frac{x+1}{x}\right)$$

as long as $x \neq 0, -1$. When $xyz = -1$, then we obtain the solutions

$$(x, y, z) = \left(x, \frac{1}{1-x}, \frac{x-1}{x}\right) \ .$$

Thus, $xyz = 1$ or $xyz = -1$.

Solution 2. We have that $xy + 1 = yt$ and $yz + 1 = zt$, so that $xyz + z = yzt = zt^2 - t$, whence $z(t^2 - 1) = xyz + t$. Similarly, $y(t^2 - 1) = x(t^2 - 1) = xyz + t$. If $x \neq y$, since $(x - y)(t^2 - 1) = 0$, we must have that $t = \pm 1$. We find that $(x, y, z, t) = ((1 - z)^{-1}, z^{-1}(z - 1), z, 1)$ and $xyz = -1$ or $(x, y, z, t) = (-(z + 1)^{-1}, -z^{-1}(z + 1), z, -1)$ and $xyz = 1$. Thus xyz is equal to 1 or -1.

Solution 3. We have that $y = 1/(t - x)$ and $z = t - (1/x) = (xt - 1)/x$. This leads to

$$\frac{1}{t - x} + \frac{x}{xt - 1} = t \Longrightarrow 0 = xt^3 - (1 + x^2)t^2 - xt + (1 + x^2) = (t^2 - 1)[xt - (1 + x^2)] = 0 \; .$$

Similarly,

$$0 = (t^2 - 1)[yt - (1 + y^2)] = (t^2 - 1)[zt - (1 + z^2)] \; .$$

Either $t^2 = 1$ or x, y, z are the roots of the quadratic equation $\lambda^2 - t\lambda + 1 = 0$. Since a quadratic has at most two roots, two of x and y must be equal, say $x = y$. But then $y = z$ contrary to hypothesis. Hence $t^2 = 1$.

Multiplying the three equations together yields that

$$t^3 = xyz + 3t + \frac{1}{xyz}$$

from which

$$0 = (xyz)^2 + (3t - t^3)(xyz) + 1 = (xyz)^2 + (3t - t)(xyz) + t^2 = (xyz + t)^2 \; .$$

Hence $xyz = t$. As in the previous solutions, we check that $t = 1$ and $t = -1$ are both possible.

420. Two circle intersect at A and B. Let P be a point on one of the circles. Suppose that PA meets the second circle again at C and PB meets the second circle again at D. For what position of P is the length of the segment CD maximum?

Solution 1. The segment CD always has the same length. The strategy is to show that the angle subtended by CD on its circle is equal to the sum of the angles subtended by AB on the two circles, and so is constant. There re a number of configurations possible. Note that (i) and (ii) do not occur with the same pair of circle. The strategy is to show that the angle subtended by CD on its circle is equal to the sum or difference of the angles subtended by AB on its two circles, and so CD is constant.

(i) A is between P and C; B is between P and D;

(ii) C is between P and A; D is between P and B;

(iii) A is between P and C; D is between P and B;

(iv) C is between P and A; B is between P and D;

(v) P is between A and C and also between B and D.

Ad (i), $\angle CBD = \angle PCB + \angle BPC = \angle ACB + \angle APB$. Ad (ii), $\angle CBD = \angle ACB - \angle APB$. Ad (iii), by the angle sum of a triangle, $\angle CAD = 180° - \angle CBD = \angle BCA + \angle BPA$. Since $ADBC$ is concyclic, $\angle CBD = \angle PAD = 180° - \angle APD - \angle ADP = \angle ADB - \angle APB$. Case (iv) is similar to (iii). Ad (v), $\angle DBC = \angle DPC - \angle PCB = \angle APB - \angle ACB$. The angle subtended by CD on the arc opposite P is $180° - \angle DBC = \angle ACB + (180° - \angle APB)$. Also, $\angle DBC = \angle APB - \angle ADB = (180° - \angle ADB) - (180° - \angle APB)$.

Solution 2. We have the same set of cases as in the first solution. Let U be the centre of the circle PAB and V the centre of the circle $ABDC$. Let UV and AB intersect in O; note that $UV \perp AB$. It is straightforward to show that triangles PAB and PDC are similar, whence $CD : AB = PC : PB$ and that triangles PBC and UBV are similar, whence $PC : PB = UV : UB$. Therefore, $CD : AB = UV : UB$ and the results follows.

421. Let $ABCD$ be a tetrahedron. Prove that

$$|AB| \cdot |CD| + |AC| \cdot |BD| \geq |AD| \cdot |BC| \ .$$

Solution 1. First, we establish a small proposition. Let $\mathbf{u}$ and $\mathbf{v}$ be any unit vectors in space and p and q any scalars. Then

$$|p\mathbf{u} + q\mathbf{v}| = |p\mathbf{v} + q\mathbf{u}| \ .$$

This is intuitively obvious, but can be formally established as follows:

$$\begin{aligned} |p\mathbf{u} + q\mathbf{v}|^2 &= (p\mathbf{u} + q\mathbf{v}) \cdot (p\mathbf{u} + q\mathbf{v}) = p^2 + q^2 + 2pq\mathbf{u} \cdot \mathbf{v} \\ &= (p\mathbf{v} + q\mathbf{u}) \cdot (p\mathbf{v} + q\mathbf{u}) = |p\mathbf{v} + q\mathbf{u}|^2 \ . \end{aligned}$$

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors and b, c, d be positive scalars for which $\overrightarrow{AB} = b\mathbf{u}$, $\overrightarrow{AC} = c\mathbf{v}$ and $\overrightarrow{AD} = d\mathbf{w}$. Thus $\overrightarrow{BC} = c\mathbf{v} - b\mathbf{u}$, $\overrightarrow{CD} = d\mathbf{w} - c\mathbf{v}$ and $\overrightarrow{BD} = d\mathbf{w} - b\mathbf{u}$.

Then

$$\begin{aligned} |AB||CD| + |AC||BD| &= b|d\mathbf{w} - c\mathbf{v}| + c|d\mathbf{w} - b\mathbf{u}| \\ &= b|d\mathbf{v} - c\mathbf{w}| + c|b\mathbf{w} - d\mathbf{u}| = |bd\mathbf{v} - bc\mathbf{w}| + |cb\mathbf{w} - cd\mathbf{u}| \\ &\geq |bd\mathbf{v} - cd\mathbf{u}| = d|b\mathbf{v} - c\mathbf{u}| = d|c\mathbf{v} - b\mathbf{u}| = |AD||BC| \ , \end{aligned}$$

as required.

Solution 2. Consider the planes of ABC and DBC as being hinged along BC. If we flatten the tetrahedron by spreading the planes apart to a dihedral angle of $180°$, then D moves to a position D' relative to A and $|AD'| \geq |AD|$. The other distances between pairs of points remain the same. It is, thus, enough to establish the result when A, B, C, D are coplanar. Suppose this to be the case.

Let a, b, c, d be complex numbers representing respectively the four points A, B, C, D. Then

$$\begin{aligned} |AB||CD| + |AC||BD| &= |(a - b)(c - d)| + |(c - a)(b - d)| \\ &\geq |(a - b)(c - d) + (c - a)(b - d)| = |(a - d)(c - b)| = |AD||BC| \ . \end{aligned}$$

(The result in the plane is known as Ptolemy's Inequality.)

Solution 3. [Q. Ho Phu] On the ray AC determine C' so that $|AC||AC'| = |AB|^2$; on the ray AD determine D' so that $|AD||AD'| = |AB|^2$. Since $AB : AC = AC' : AB$ and angle A is common, triangles ABC and $AC'B$ are similar, whence $BC' : BC = AB : AC$ and

$$|BC'| = \frac{|BC||AB|}{|AC|} = \frac{|BC||AD||AB|}{|AC||AD|} \ .$$

Similarly,

$$|BD'| = \frac{|BD||AB|}{|AD|} = \frac{|BD||AC||AB|}{|AD||AC|} \ ,$$

and

$$|C'D'| = \frac{|CD||AD'|}{|AC|} = \frac{|CD||AB|^2}{|AD||AC|} \ .$$

In the triangle $BC'D'$, we have that $|BD'| + |C'D'| > |BC'|$, whence

$$|BD||AC| + |CD||AB| > |AD||BC|$$

as desired.

422. Determine the smallest two positive integers n for which the numbers in the set $\{1, 2, \cdots, 3n-1, 3n\}$ can be partitioned into n disjoint triples $\{x, y, z\}$ for which $x + y = 3z$.

Solution. Suppose that the partition consists of the triples $[x_k, y_k, z_k]$ $(1 \leq k \leq n)$. Then

$$\sum_{i=1}^{3n} i = \sum_{k=1}^{n} (x_k + y_k + z_k) = 4 \sum_{k=1}^{n} z_k$$

so that 4 must divide $\frac{1}{2}3n(3n+1)$, or that $3n(3n+1)$ is a multiple of 8. Thus, either $n \equiv 0$ or $n \equiv 5$ (mod 8).

$n = 5$ is possible. Here are some examples:

$$[1, 11, 4], [2, 13, 5], [3, 15, 6], [9, 12, 7], [10, 14, 8]$$

$$[1, 14, 5], [2, 10, 4], [3, 15, 6], [9, 12, 7], [11, 13, 8]$$

$$[1, 8, 3], [2, 13, 5], [12, 15, 9], [4, 14, 6], [10, 11, 7]$$

$$[1, 11, 4], [2, 7, 3], [5, 13, 6], [10, 14, 8], [12, 15, 9]$$

$$[1, 8, 3], [2, 13, 5], [4, 14, 6], [10, 11, 7], [12, 15, 9]$$

Adjoining to any of these solutions the eight triples

$$[19, 29, 16], [21, 30, 17], [26, 28, 18], [27, 33, 20], [31, 35, 22], [32, 37, 23], [34, 38, 24], [36, 39, 25]$$

yields a possibility for $n = 13$.

For $n = 8$, we have

$$[1, 5, 2], [3, 9, 4], [6, 18, 8], [7, 23, 10], [14, 19, 11], [16, 20, 12], [17, 22, 13], [21, 24, 15]$$

There are many other possibilities.