International Mathematical Talent Search – Round 15

Problem 1/15. Is it possible to pair off the positive integers $1, 2, 3, \ldots, 50$ in such a manner that the sum of each pair of numbers is a different prime number?

Problem 2/15. Substitute different digits $(0, 1, 2, \ldots, 9)$ for different letters in the following alphametics to ensure that the corresponding additions are correct. (The two problems are independent of one another.)

$$\begin{array}{cccccccc} & H & A & R & R & I & E & T \\ & M & A & R & R & I & E & D \\ + & & & & & H & E & R \\ \hline & D & E & N & T & I & S & T \end{array} \qquad \begin{array}{cccccc} & D & I & A & N & A \\ & & & A & N & D \\ & S & A & R & A & H \\ + & & & A & R & E \\ \hline R & E & B & E & L & S \end{array}$$

Problem 3/15. Two pyramids share a seven-sided common base, with vertices labeled as A_1, A_2, A_3, $\ldots$, A_7, but they have different apexes, B and C. No three of these nine points are colinear. Each of the 14 edges BA_i and CA_i ($i = 1, 2, \ldots, 7$), the 14 diagonals of the common base, and the segment BC are colored either red or blue. Prove that there are three segments among them, all of the same color, that form a triangle.

Problem 4/15. Suppose that for positive integers a, b, c and x, y, z, the equations $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$ are satisfied. Prove that

$$(a + x)^2 + (b + y)^2 \leq (c + z)^2,$$

and determine when equality holds.

Problem 5/15. Let C_1 and C_2 be two circles intersecting at the points A and B, and let C_0 be a circle through A, with center at B. Determine, with proof, conditions under which the common chord of C_0 and C_1 is tangent to C_2?