Canadian Open Mathematics Challenge 2021

Official Solutions

A competition of the Canadian Mathematical Society.

Revised January 25, 2022

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2021/practice.html

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Part A

A1 Let x be a real number such that $(x-2)(x+2) = 2021$. Determine the value of $(x-1)(x+1)$.

Solution: Expanding $(x-2)(x+2) = 2021$ yields $x^2 - 4 = 2021$. Hence, $x^2 = 2025$. Therefore, $(x-1)(x+1) = x^2 - 1 = 2025 - 1 = 2024$.

Answer: $\boxed{2024}$.

A2. Julia had eaten a box of candies in 4 days. On the first day she ate $\frac{1}{5}$ of the total number of candies. On the second day she ate half of what was left after the first day. On the third day she ate half of what was left after the second day. What portion of the candies initially contained in the box did she eat on the fourth day? Express your answer as an irreducible fraction.

Solution 1: We organize the information in the form of a table.

day #	1	2	3	4
ate of the total	$\frac{1}{5}$	$\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$	$\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$	$\frac{1}{5}$
left of the total	$1 - \frac{1}{5} = \frac{4}{5}$	$\frac{4}{5} - \frac{2}{5} = \frac{2}{5}$	$\frac{2}{5} - \frac{1}{5} = \frac{1}{5}$	0

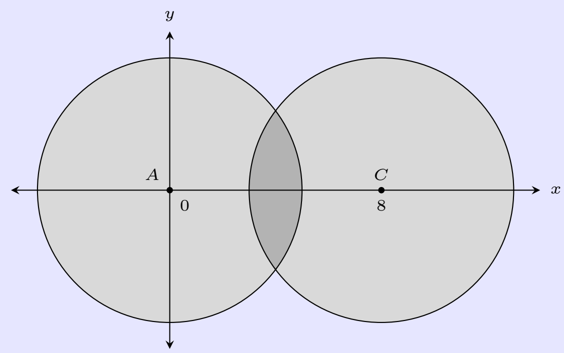
Solution 2: Assume that box contains $5n$ (for some $n \geq 1$) candies and count the numbers of candies (instead of fractions of the box).

day #	1	2	3	4
ate	n	$\frac{1}{2} \times 4n = 2n$	$\frac{1}{2} \times 2n = n$	n
left	$5n - n = 4n$	$4n - 2n = 2n$	$2n - n = n$	0

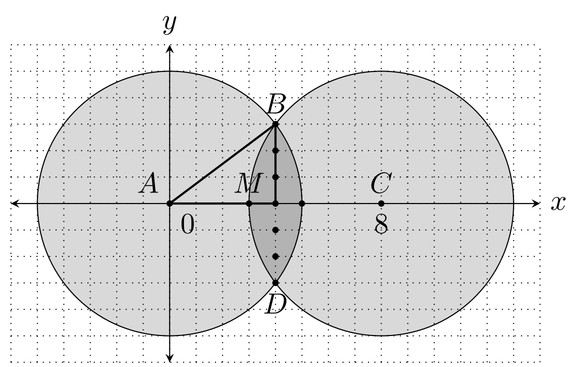
The portion she ate on the 4th day is $\frac{n}{5n} = \frac{1}{5}$.

Answer: $\boxed{1/5}$.

A3 Two circles, each of radius 5 units long, are drawn in the coordinate plane such that their centres A and C have coordinates $(0,0)$ and $(8,0)$ respectively. How many points where both coordinates are integers are within the intersection of these circles (including its boundary)?



Solution: Let M be the midpoint of AC and the circumferences of the circles intersect at points B and D. Then AM=4 units and AMB is a right triangle with AB=5.



Nowing the $3-4-5$ Pythagorean triple, MB=3 units. Therefore, there are 7 lattice points on BD (including the endpoints). There are two more points on the boundary of the intersection: they have coordinates $(3,0)$ and $(5,0)$. The total is 9 lattice points.

Answer: $\boxed{9}$.

A4. Marija travels to school by a combination of walking and skateboarding. She can get there in 38 minutes if she walks for 25 minutes and skateboards for 13, or in 31 minutes if she walks for 11 and skateboards for 20. How long (in minutes) would it take her to walk to school?

Solution 1: Denoting her walking speed by w, skateboarding speed by s, and distance to school by D, we have

$$D = 25w + 13s, \qquad \text{and } D = 11w + 20s.$$

Subtracting these gives $7s = 14w$, and so $D = 51w$. Hence, it will take 51 minutes.

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Solution 2: Decreasing the skateboarding time by 7 minutes increases the walking time by $14 = 2 \times 7$ minutes and the total time taken by 7 minutes. So decreasing the skateboarding time by 13 minutes will increase the walking time by $26 = 2 \times 13$ minutes and the total time by 13 minutes, from 38 to 51.

Answer: $\boxed{51}$.

Part B

B1. A bag contains two regularly shaped (cubic) dice which are identical in size. One die has the number 2 on every side. The other die has the numbers 2 on three sides and number 4 on each side opposite to one that has number 2. You pick up a die and look at one side of it, observing the number 2. What is the probability the opposite side of the die has the number 2 as well? Express your answer as an irreducible fraction.

Solution: There are 9 different sides we could be looking at: either side of the first dice, or 3 sides of the second dice. In 6 of those possibilities (i.e. for each side of the first dice), the other side is also a 2, so the probability is $\frac{6}{9} = \frac{2}{3}$.

Answer: $\boxed{2/3}$.

B2. When the product

$$(2021x^{2021} + 2020x^{2020} + \cdots + 3x^3 + 2x^2 + x)(x^{2021} - x^{2020} + \cdots + x^3 - x^2 + x - 1)$$

is expanded and simplified, what is the coefficient of x^{2021}?

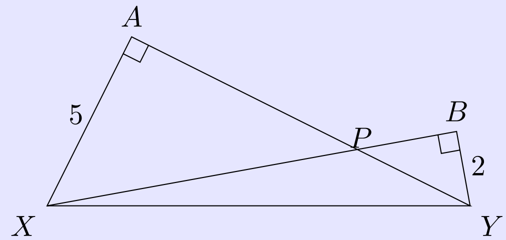
Solution: Expanding terms that yield x^{2021} we get

$$\begin{aligned}
(2021x^{2021})(-1) + (2020x^{2020})(x) + (2019x^{2019})(-x^2) + \cdots = \\
&= -2021x^{2021} + 2020x^{2021} - 2019x^{2021} + \cdots \\
&= (-2021 + [2020 - 2019] + [2018 - 2017] + \cdots + [4 - 3] + [2 - 1])x^{2021} \\
&= (-2021 + 1010 \times 1)x^{2021} \\
&= -1011x^{2021}
\end{aligned}$$

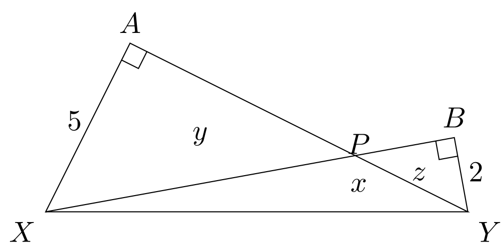
$\therefore$ The coefficient of x^{2021} is -1011.

Answer: $\boxed{-1011}$.

B3. Two right triangles ΔAXY and ΔBXY have a common hypotenuse XY and side lengths (in units) $AX = 5$, $AY = 10$, and $BY = 2$. Sides AY and BX intersect at P. Determine the area (in square units) of ΔPXY.



Solution 1:



From the right triangle AXY we find $XY^2 = XA^2 + AY^2 = 125$. Then from the right triangle BXY, $BX = \sqrt{XY^2 - BY^2} = 11$.

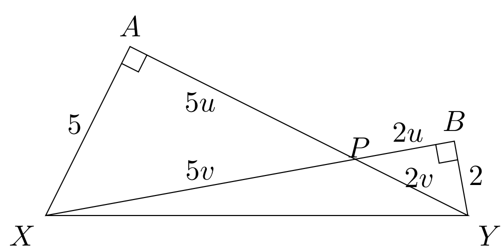
Labelling by x, y and z the area of the triangles PXY, AXP and BPY respectively (see the diagram), we get

$$x + y = \frac{1}{2}(5)(10) = 25$$
$$x + z = \frac{1}{2}(2)(11) = 11$$

As ΔPAX and ΔPBY are right angled and $\angle APX = \angle BPY$, the triangles are similar. Then, since $\frac{BY}{AX} = \frac{2}{5}$ we can deduce that $z = \left(\frac{2}{5}\right)^2 y = \frac{4}{25}y$. Substituting $z = \frac{4}{25}y$ into the system of equations and solving it gives $x = 25/3$ and $y = 50/3$. Thus, the desired area is $\frac{25}{3}$ square units.

Answer: $\boxed{25/3}$.

Solution 2:



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From the right triangle AXY we find $XY^2 = XA^2 + AY^2 = 125$. Then from the right triangle BXY, $BX = \sqrt{XY^2 - BY^2} = 11$.

As ΔPAX and ΔPBY are right angled and $\angle APX = \angle BPY$, the triangles are similar. We have $AP : PB = XP : PY = 5 : 2$, so $AP = 5u$, $PB = 2u$, $XP = 5v$, $PY = 2v$, for some u and v. Thus,

$$BX = 2u + 5v = 11$$
$$AY = 5u + 2v = 10$$

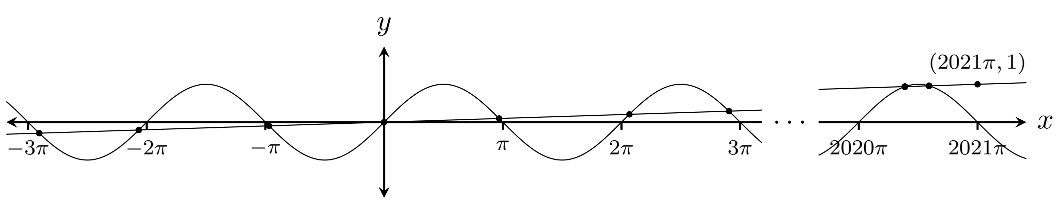
Solving this system we obtain $u = \frac{4}{3}$ and $v = \frac{5}{3}$. Thus, $XP = \frac{25}{3}$.

Therefore, the desired area is $\frac{1}{2} XP \cdot BY = \frac{25}{3}$ square units.

Answer: $\boxed{25/3}$.

B4. The equation $\sin x = \frac{x}{2021\pi}$ has exactly n solutions. Find n.

Solution: Notice that $x = 0$ is a solution. As well, both functions are odd, so due to the symmetry if x^* is a solution then $-x^*$ is also a solution.



Since $-1 \le \sin x \le 1$, all solutions are in the interval for which $-1 \le \frac{x}{2021\pi} \le 1$, that is $[-2021\pi, 2021\pi]$.

The function $y = \sin x$ is 2π-periodic. There are 1010 periods and a half of a period on the segment $(0, 2021\pi]$. There are 2 intersections of the line $y = \frac{x}{2021\pi}$ and the sine on each period except the first one $(0, 2\pi]$, where we have only one intersection. There are also two intersections on the last half of the period. Consequently, there are $1 + 2020 = 2021$ solutions on $(0, 2021\pi]$. Similarly, there are 2021 solutions on $[-2021\pi, 0)$. In total, there are $n = 1 + 2 \times 2021 = 4043$ solutions.

Answer: $\boxed{4043}$.

Part C

C1.

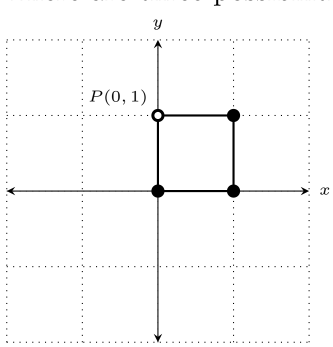
a. Determine all points $P(x, y)$ such that $(0, 0)$, $(1, 1)$, $(1, 0)$ and P are vertices of a parallelogram.

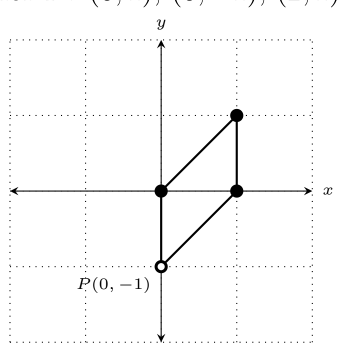
b. Two parallel lines intersect the (horizontal) parabola $x = y^2$ at four distinct points: $(0, 0)$, $(1, 1)$, $(9, 3)$ and Q. Determine all possible coordinates of the point Q.

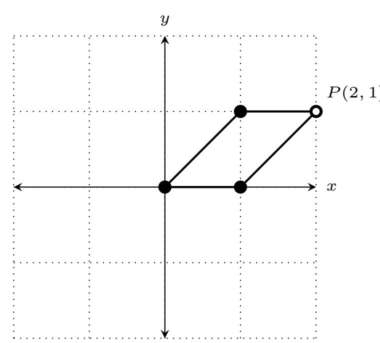
c. Two parallel lines intersect the parabola $x = y^2$ at four distinct points: $(0, 0)$, $(1, 1)$, (a^2, a) and V. Here $a \neq 0, \pm 1$ is a real number. Determine all possible coordinates of the point V. The answer should be expresed in term of a.

Solution:

a. There are three possibilities for P: $(0, 1)$, $(0, -1)$, $(2, 1)$.

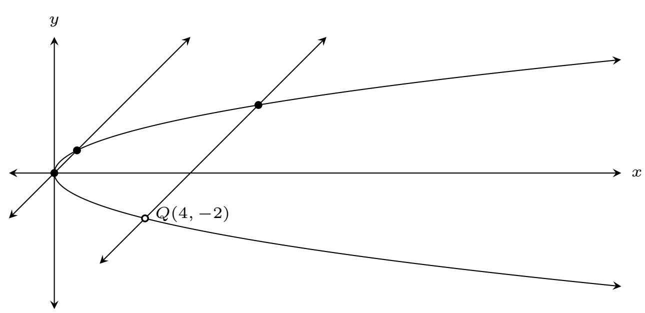






b. There are three possible pairs of parallel lines - the line through $(0, 0)$ can go through $(1, 1)$, or $(9, 3)$ or Q.

- The line through $(0, 0)$ and $(1, 1)$ is parallel to $y = x - 6$ through $(9, 3)$. The solutions to $y + 6 = y^2$ give the y coordinates of the intersection points as $x = y + 6$ and $x = y^2$. Hence at $y = 3$ or $y = -2$, so the y coordinate of Q here is -2 and then $x = 4$.



- The line through $(0, 0)$ and $(9, 3)$ is parallel to $y = \frac{x+2}{3}$ through $(1, 1)$. The solutions to $3y - 2 = y^2$, are $y = 1$ or $y = 2$, so the y coordinate of Q here is 2 and $x = 4$.

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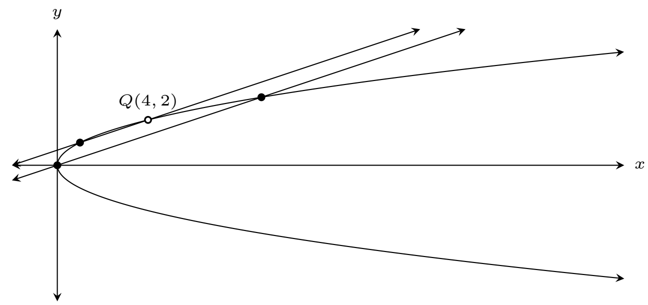
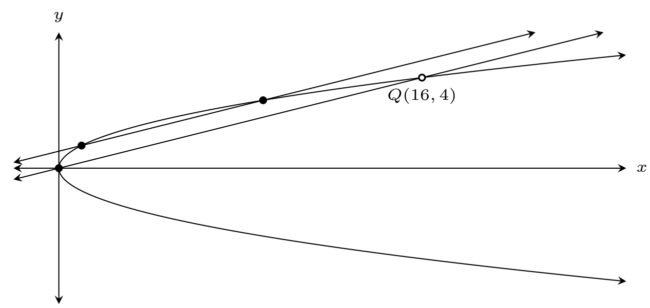


- The line through $(1,1)$ and $(9,3)$ is parallel to $y = \frac{x}{4}$ through $(0,0)$. The non-zero solution to $4y = y^2$ is $y = 4$, so the y coordinate of Q here is 4 and $x = 16$.



Thus, Q could be at $(4,-2)$, $(4,2)$, $(16,4)$.

c. There are three possible pairs of parallel lines - the line through $(0,0)$ can go through $(1,1)$, or (a^2,a) or V.

- The line through $(0,0)$ and $(1,1)$ is parallel to $y = x + a - a^2$ through (a^2,a). The solutions to $y + a(a-1) = y^2$ give the y coordinates of the intersection points, hence at $y = a$ or $y = 1-a$, so the y coordinate of V here is $1-a$ and then $x = (1-a)^2$.
- The line through $(0,0)$ and (a^2,a) is parallel to $y = \frac{x+a-1}{a}$ through $(1,1)$. The solutions to $ay + 1 - a = y^2$, are $y = 1$ or $y = a-1$, so the y coordinate of V here is $a-1$ and $x = (a-1)^2$.
- The line through $(1,1)$ and (a^2,a) is parallel to $y = \frac{x}{a+1}$ through $(0,0)$. The non-zero solution to $(a+1)y = y^2$ is $y = a+1$, so the y coordinate of V here is $a+1$ and $x = (a+1)^2$.

Thus, V could be at $((a-1)^2, \pm(a-1)$ and $((a+1)^2, a+1))$.

A variation of the solution for C1(c): There are three possible pairs of parallel lines - the line through $(0,0)$ can go through $(1,1)$, or (a^2,a) or V.

- The line through $(0,0)$ and $(1,1)$ has slope 1. A line through (a^2,a) parallel to it has the equation $\frac{y-a}{x-a^2} = 1$. Substitute $x = y^2$ in order to find the points of intersection of the line with the parabola. We have $\frac{y-a}{y^2-a^2} = 1$ or $\frac{y-a}{(y-a)(y+a)} = 1$. To exclude the original point we assume $y \neq a$. Thus, we can cancel $y - a$ and get $y + a = 1$, so $y = 1 - a$. Then $x = (1-a)^2$.

- The line through $(0,0)$ and (a^2,a) has slope $\frac{1}{a}$. A line through $(1,1)$ parallel to it has the equation $\frac{y-1}{x-1}=\frac{1}{a}$. Substitute $x=y^2$ in order to find the points of intersection of the line with the parabola. We have $\frac{y-1}{y^2-1}=\frac{1}{a}$ or $\frac{y-1}{(y-1)(y+1)}=\frac{1}{a}$. To exclude the original point we assume $y\neq 1$. Thus, we can cancel $y-1$ and get $y+1=a$, so $y=a-1$. Then $x=(a-1)^2$.
- The line through $(1,1)$ and (a^2,a), $a\neq\pm1$ has slope $\frac{a-1}{a^2-1}=\frac{1}{a+1}$. A line through $(0,0)$ parallel to it has the equation $\frac{y}{x}=\frac{1}{a+1}$. Substitute $x=y^2$ in order to find the points of intersection of the line with the parabola. We have $\frac{y}{y^2}=\frac{1}{a+1}$. To exclude the original point we assume $y\neq 0$. Thus, we can cancel y and get $y=a+1$. Then $x=(a+1)^2$.

Thus, V could be at $((a-1)^2,\pm(a-1)$ and $((a+1)^2,a+1)$. Finally, we should ensure that none of the points $(0,0),(1,1),(a^2,a)$ and V coincide. The problem poses the restriction $a\neq 0,\pm1$. In addition, we find that for $a=\frac{1}{2}$, $((a-1)^2,-(a-1))=(a^2,a)=(\frac{1}{4},-\frac{1}{2})$, and for $a=2$, $((a-1)^2,(a-1))=(1,1)$. Therefore the answer is: for $a\neq 0,\pm1,\frac{1}{2},2$ point V could be at $((a-1)^2,\pm(a-1))$ and $((a+1)^2,a+1)$; for $a=\frac{1}{2}$ point V could be at $(\frac{9}{4},\frac{3}{2})$ or $(\frac{1}{4},-\frac{1}{2})$; for $a=2$ point V could be at $(1,-1)$ or $(9,3)$. We also note that for $a=-1$, point V could be at $(4,-2)$ or $(4,2)$.

C2. Let $m,n\geq 2$ be positive integers. Each entry of an $m\times n$ grid contains a real number in the range $[-1,1]$, i.e. between -1 and 1 inclusively. The grid also has the property that the sum of the four entries in every 2×2 *subgrid* is equal to 0. (A 2×2 subgrid is the intersection of two adjacent rows and two adjacent columns of the original grid.)
Let S be the sum of all of the entries in the grid.

a. Suppose $m=6$ and $n=6$. Explain why $S=0$.

b. Suppose $m=3$ and $n=3$. If the elements of the grid are

a	b	c
d	e	f
g	h	i

,

show that $S+e=a+i=c+g$.

c. Suppose $m=7$ and $n=7$. Determine the maximum possible value of S.

Solution:

a. A 6×6 grid can be partitioned into 2×2 subgrids, because there is an even number of squares along each side of the grid. Since the sum of the entries of each of the subgrids is 0, the sum of all of the entries of the grid is 0. Therefore, $S=0$.

b. We have $S=a+b+c+d+e+f+g+h+i$ and the sum of the four entries in every 2×2 subgrid is equal to 0. Thus,

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$$S+e=(b+c+e+f)+(d+e+g+h)+a+i=0+0+a+i=a+i.$$
$$S+e=(a+b+d+e)+(e+f+h+i)+c+g=0+0+c+g=c+g.$$

c. The sum of all of the entries of the grid can be determined more simply with the following fact.

Fact: In the following 3×3 grid, where $a+b+d+e=e+f+h+i=0$,

a	b	c
d	e	f
0	h	i

the sum of the entries is $S=c-e$. The proof is similar to (b), setting $g=0$:

$$S+e=(a+b+d+e)+(e+f+h+i)+c=0+0+c=c.$$

We can partition the 7×7 grid as follows:

$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_4	b_4	c_4
$\cdots$	$\cdots$	$\cdots$	$\cdots$	d_4	e_4	f_4
$\cdots$	$\cdots$	a_3	b_3	c_3	g_4	h_4
$\cdots$	$\cdots$	d_3	e_3	f_3	$\cdots$	$\cdots$
a_2	b_2	c_2	g_3	h_3	$\cdots$	$\cdots$
d_2	e_2	f_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$
c_1	g_2	h_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The entries not labeled can be broken down into 2×2 grids, so their sum is 0.

From (b) and the fact we used above, the sums of elements in the overlapping 3×3 grids

a_2	b_2	c_2
d_2	e_2	f_2
c_1	g_2	h_2

,

a_3	b_3	c_3
d_3	e_3	f_3
0	g_3	h_3

,

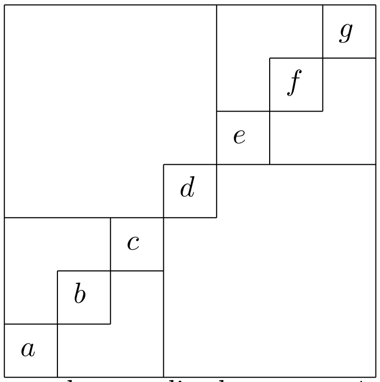
a_4	b_4	c_4
d_4	e_4	f_4
0	g_4	h_4

are
$c_1-e_2+c_2$, c_3-e_3 and c_4-e_4 respectively.

Hence, the sum of all of the entries of the 7×7 grid is

$$c_1-e_2+c_2-e_3+c_3-e_4+c_4.$$

An alternative way to show that S in a 7×7 grid can be expressed as the alternating sum of the diagonal elements is as follows.



The 7×7 square is covered by two 3×3 squares (with diagonal elements a, b, c and e, f, g respectively) and by two 4×4 squares (overlapping in one element d). By part (b), the 3×3 squares have element sums of $a-b+c$ and $e-f+g$ respectively. The 4×4 squares can each be partitioned into four 2×2 squares and thus each has element sum 0. Thus we get

$$S = a-b+c+e-f+g+0+0-d = a-b+c-d+e-f+g.$$

Since each entry lies between -1 and 1 inclusively, this sum is at most 7. Hence, the sum of all the entries is at most 7. This maximum sum can be obtained as follows:

1	0	1	0	1	0	1
0	−1	0	−1	0	−1	0
1	0	1	0	1	0	1
0	−1	0	−1	0	−1	0
1	0	1	0	1	0	1
0	−1	0	−1	0	−1	0
1	0	1	0	1	0	1

or

1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1

Answer: $\boxed{7}$.

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C3. Xintong plays a game of turning one six-digit number into another. The numbers can have leading zeros, but cannot go over 6 digits or below 0. He can only make the following moves, any number of times in any order:

- R: rotate the last digit to the start, for example, 092347 $\to$ 709234, or
- A: add 1001 to the number, for example, 709234 $\to$ 710235, or
- S: subtract 1001 from the number, for example, 709234 $\to$ 708233.

a. Show that it is possible to turn 202122 into 313233.

b. Show that turning 999999 into 000000 can be done in eight moves.

c. Show that any multiple of 11 remains a multiple of 11 after any sequence of moves.

d. Show that it is impossible to turn 112233 into 000000.

Solution:

a. We need to add one to every digit, so any combination of three additions to distinct digits and six rotations will work. For example, it can be reached by $A, R, A, R, A, R, R, R, R$.

b. This can be done through moves: S, R, A, R, R, A, R, S.

c. Since 1001 is divisible by 11, adding (or subtracting) 1001 to a number divisible by 11 will result in a number divisible by 11. If n has digits $a_1a_2a_3a_4a_5a_6$, then n is divisible by 11 if and only if $a_1 - a_2 + a_3 - a_4 + a_5 - a_6$ is. Rotating makes the sum negative, so divisibility by 11 is preserved.

   Alternative proof of the implication $11|n \to 11|R(n)$*:* Let $n = 10^5a + 10^4b + 10^3c + 10^2d + 10e + f$. Then after rotation we have $R(n) = 10^5f + 10^4a + 10^3b + 10^2c + 10d + e$. Consider the sum

$$n + R(n) = 11 \cdot 10^4a + 11 \cdot 10^3b + 11 \cdot 10^2c + 11 \cdot 10d + 11e + 100001f.$$

   Note that 100001 is divisible by 11. Since both the sum $n + R(n)$ and n are divisible by 11, we conclude that $R(n)$ is divisible by 11.

d. We can show that (non-)divisibility by 7, or 11, or 13 is an invariant. If n is not divisible by 7 or 11 or 13, then the same is true for $n \pm 1001$, since $7 \cdot 11 \cdot 13 = 1001$.

   Now, let $n = abcdef$ and $R(n) = fabcde$. Then

$$R(n) = \frac{n-f}{10} + f \times 10^5.$$

   Therefore $10R(n) = n - f + f \times 10^6$.

   Since $10^6 = 999 \times 1001 + 1$, we write $10^6 \equiv 1(\text{mod } 1001)$. So, we conclude

$$10R(n) \equiv n(\text{mod } 1001).$$

   In other words, n and $10R(n)$ are either both divisible by 1001 or give the same remainder when divided by 1001. So, if n is not divisible by 1001 then $R(n)$ must be not divisible by 1001.

We observe that 112233 is not divisible by 7 and thus not divisible by 1001, while 000000 is divisible by 1001. Thus, the transfer of 112233 to 000000 by a sequence of R, S and A is impossible.

Alternative solution for C3(d): First, observe that we can turn 112233 into 000000 if and only if we can turn 000000 into 112233. This is because each move can be inverted: the inverse of A is S, and the inverse of R is five consecutive R moves.

Next, let T be the set of all numbers of the form $abcabc$. Since T contains 000000 but not 112233, it suffices to prove that if you start in T, then you must stay in T.

This is obvious for the R move: $abcabc$ becomes $cabcab$.

For the other moves, observe that T is exactly the set of numbers between 0 and 999999 that are multiples of 1001 (since $abcabc = (abc) \times 1001$). Therefore A and S moves must take elements of T to other elements of T.

C4. We call (F, c) a *good pair* if the following three conditions are satisfied:
(1) $F(x) = a_0 + a_1 x + \cdots a_m x^m$, $(m \geq 1)$ is a *non-constant* polynomial with integer coefficients.
(2) c is a real number that is not an integer.
(3) $F(c)$ is an integer.
For example, both $(6x, \frac{1}{3})$ and $(1 + x^3, 5^{1/3})$ are good pairs, but none of the following pairs $(6x, \frac{1}{4})$, $(6x, 2)$, $(\frac{x}{6}, \frac{1}{3})$, $(\frac{x^2}{6}, 6)$ is good.

a. Let $c = \frac{1}{2}$. Give an example of F such that (F, c) is a good pair but $(F, c+1)$ is not.

b. Let $c = \sqrt{2}$. Give an example of F such that both (F, c) and $(F, c+1)$ are good pairs.

c. Show that for any good pair (F, c), if c is rational then there exists infinitely many non-zero integers n such that $(F, c+n)$ is also a good pair.

d. Show that if $(F, c+n)$ is a good pair for every integer n, then c is rational.

Solution:

a. Let $F(x) = 2x^3 + x^2 + x$. Then $F(\frac{1}{2}) = 1$ and $F(\frac{3}{2}) = \frac{21}{2}$.

b. Let $F(x) = (x^2 - 2)((x-1)^2 - 2))$. Then $F(\sqrt{2}) = F(\sqrt{2} + 1) = 0$.

c. Let p and q be integers such that $c = \frac{p}{q}$ is not an integer and let $F(x) = \sum_{i=0}^{m} a_i x^i$ with integer coefficients a_i.

Then, $F(c+n) - F(c) = \sum_{i=0}^{m} a_i \left((c+n)^i - c^i\right)$.

For all $1 \leq i \leq m$ we have

$$(c+n)^i - c^i = \sum_{j=0}^{i} \binom{i}{j} c^j n^{i-j} - c^i = n \sum_{j=0}^{i-1} \binom{i}{j} c^j n^{i-1-j} = n \sum_{j=0}^{i-1} \binom{i}{j} \left(\frac{p}{q}\right)^j n^{i-1-j}.$$

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If now we choose $n = q^k$, $k \geq m - 1$ then each of these differences is an integer. So there are infinitely many choices for n.

d. Assume otherwise; that is, there is some irrational number c such that $F(c+n)$ is an integer for every integer n. Take such a polynomial F of minimal degree. It is clear that F is not linear. We can take a polynomial $G(x) = F(x+1) - F(x)$, and now $G(x)$ has degree one lower than F and is also an integer on $c + n$ for all integers n, which is a contradiction.

Note that if $c = \frac{p}{q}$ is rational then $F(x) = qx$ makes the required good pairs.

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