Canadian Mathematical Society
Société mathématique du Canada

The 2021 Canadian Open Mathematics Challenge

October 28 & 29, 2021

STUDENT INSTRUCTIONS

General Instructions:

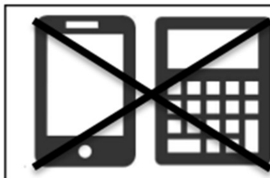
1) Do not open the exam booklet until instructed to do so by your proctor (supervising teacher).

2) **Before the exam time starts, the proctor will give you a few minutes to fill in the Participant Identification on the cover page of the exam.** You don't need to rush. Be sure to fill in all required information fields and write legibly.

3) **Readability counts:** Make sure the pencil(s) you use are dark enough to be clearly legible throughout your exam solutions.

4) Once you have completed the exam and given it to the proctor/teacher you may leave the room.

5) The questions and solutions of the COMC exam must not be publicly discussed or shared (including online) for at least 24 hours.



Mobile phones and calculators are NOT permitted.

Exam Format:

There are three parts to the COMC to be completed in a total of 2 hours and 30 minutes:

PART A: Four introductory questions worth 4 marks each. You do not have to show your work. A correct final answer gives full marks. However, if your final answer is incorrect and you have shown your work in the space provided, you might earn partial marks.

PART B: Four more challenging questions worth 6 marks each. Marking and partial marks follow the same rule as part A.

PART C: Four long-form proof problems worth 10 marks each. Complete work must be shown. Partial marks may be awarded.

Diagrams provided are *not* drawn to scale; they are intended as aids only.

Scrap paper/extra pages: You *may* use scrap paper, but you have to throw it away when you finish your work and hand in your booklet. Only the work you do on the pages provided in the booklet will be evaluated for marking. Extra pages are not permitted to be inserted in your booklet.

Exact solutions: It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as 12.566, 4.646, etc.

Awards: The names of all award winners will be published on the Canadian Mathematical Society website.

2021 Canadian Open Mathematics Challenge

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Student Identification

Please print clearly and complete all information below. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your test supervisor's signed declaration.

1. Given/First Name: (Required)

2. Sur-/Last Name: (Required)

3. Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since at least September 15th of this year? (Required for qualification)

☐ **Yes** ☐ No

4. Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of your current address)? (Required for qualification)

☐ **Yes** ☐ No

5. Date of Birth: (Required)

Y Y Y Y M M D D

6. Grade: (Required)

☐ 8 ☐ 10 ☐ 12
☐ 9 ☐ 11 ☐ Cégep
other: ______

7. Gender: (Optional)

☐ Female
☐ Male
☐ Other

8. Email Address: (Optional)

DO NOT PHOTOCOPY BLANK EXAMS!
Each page of each copy is uniquely pre-coded to facilitate computer-assisted marking.

Question A1 (4 points)

Let x be a real number such that $(x-2)(x+2) = 2021$. Determine the value of $(x-1)(x+1)$.

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A2 (4 points)

Julia had a box of candies and ate them all in 4 days. On the first day she ate $\frac{1}{5}$ of the total number of candies. On the second day she ate half of what was left after the first day. On the third day she ate half of what was left after the second day. What portion of the candies initially contained in the box did she eat on the fourth day? Express your answer as a reduced fraction.

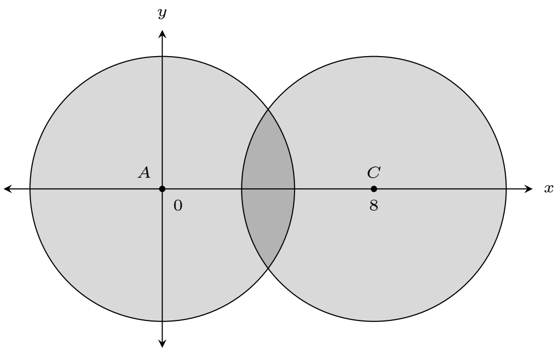
Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A3 (4 points)

Two circles, each of radius 5 units, are drawn in the coordinate plane such that their centres A and C have coordinates $(0, 0)$ and $(8, 0)$ respectively. How many points of the plane where both coordinates are integers lie within the intersection of these circles (including its boundary)?



Your final answer:

[A correct answer here earns full marks]

Question A4 (4 points)

Marija travels to school by a combination of walking and skateboarding. She can get there in 38 minutes if she walks for 25 minutes and skateboards for 13, or in 31 minutes if she walks for 11 and skateboards for 20. How long (in minutes) would it take her to walk to school?

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B1 (6 points)

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NO PHOTOCOPIES!

A bag contains two regularly shaped (cubic) dice which are identical in size. One die has the number 2 on every side. The other die has the numbers 2 on three sides and number 4 on each side opposite to one that has number 2. You pick up a die and look at one side of it, observing the number 2. What is the probability the opposite side of the die has the number 2 as well?
Express your answer as a reduced fraction.

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B2 (6 points)

When the product

$$(2021x^{2021} + 2020x^{2020} + \cdots + 3x^3 + 2x^2 + x)(x^{2021} - x^{2020} + \cdots + x^3 - x^2 + x - 1)$$

is expanded and simplified, what is the coefficient of x^{2021}? We assume that the indicated patterns continue.

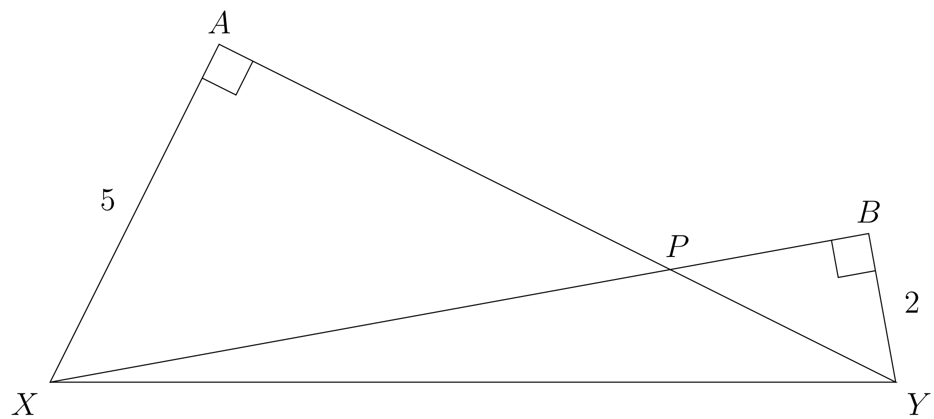
Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B3 (6 points)

Two right triangles ΔAXY and ΔBXY have a common hypotenuse XY. Sides AY and BX intersect at P. Side lengths (in units) are $AX = 5$, $AY = 10$, and $BY = 2$. Determine the area (in square units) of ΔPXY.



Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B4 (6 points)

The equation $\sin x = \dfrac{x}{2021\pi}$ has exactly n solutions. We measure x in radians. Find n.

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question C1 (10 points)

(a) Determine all points $P(x, y)$ such that $(0, 0)$, $(1, 1)$, $(1, 0)$ and P are vertices of a parallelogram.

(b) Two parallel lines intersect the (horizontal) parabola $x = y^2$ at four distinct points: $(0, 0)$, $(1, 1)$, $(9, 3)$ and Q. Determine the coordinates of all possible points $Q(x, y)$.

(c) Two parallel lines intersect the parabola $x = y^2$ at four distinct points: $(0, 0)$, $(1, 1)$, (a^2, a) and V. Here $a \neq 0, \pm 1$ is a real number. Determine the coordinates of all possible points $V(x, y)$. The answer should be expressed in terms of a.

Your solution:

You **must** show all your work.

Question C2 (10 points)

Let $m, n \geq 2$ be positive integers. Each entry of an $m \times n$ grid contains a real number in the range $[-1, 1]$, i.e. between -1 and 1 inclusively. The grid also has the property that the sum of the four entries in every 2×2 *subgrid* is equal to 0. (A 2×2 subgrid is the intersection of two adjacent rows and two adjacent columns of the original grid.)
Let S be the sum of all of the entries in the grid.

(a) Suppose $m = 6$ and $n = 6$. Explain why $S = 0$.

(b) Suppose $m = 3$ and $n = 3$. If the elements of the grid are

a	b	c
d	e	f
g	h	i

,

show that $S + e = a + i = c + g$.

(c) Suppose $m = 7$ and $n = 7$. Determine the maximum possible value of S.

Your solution:

You **must** show all your work.

Question C3 (10 points)

Xintong plays a game of turning one six-digit number into another. The numbers can have leading zeros, but cannot go over 6 digits or below 0. He can only make the following moves, any number of times in any order:

- R: rotate the last digit to the start, for example, 092347 $\to$ 709234, or
- A: add 1001 to the number, for example, 709234 $\to$ 710235, or
- S: subtract 1001 from the number, for example, 709234 $\to$ 708233.

(a) Show that it is possible to turn 202122 into 313233.

(b) Show that turning 999999 into 000000 can be done in eight moves.

(c) Show that any multiple of 11 remains a multiple of 11 after any sequence of moves.

(d) Show that it is impossible to turn 112233 into 000000.

Your solution:

You **must** show all your work.

Question C4 (10 points)

We call (F, c) a *good pair* if the following three conditions are satisfied:

(1) $F(x) = a_0 + a_1 x + \cdots a_m x^m$, $(m \geq 1)$ is a nonconstant polynomial with integer coefficients.

(2) c is a real number that is not an integer.

(3) $F(c)$ is an integer.

For example, both $(6x, \frac{1}{3})$ and $(1 + x^3, 5^{1/3})$ are good pairs, but none of the following pairs $(6x, \frac{1}{4})$, $(6x, 2)$, $(\frac{x}{6}, \frac{1}{3})$, $(\frac{x^2}{6}, 6)$ is good.

(a) Let $c = \frac{1}{2}$. Give an example of F such that (F, c) is a good pair but $(F, c+1)$ is not.

(b) Let $c = \sqrt{2}$. Give an example of F such that both (F, c) and $(F, c+1)$ are good pairs.

(c) Show that for any good pair (F, c), if c is rational then there exists infinitely many non-zero integers n such that $(F, c+n)$ is also a good pair.

(d) Show that if $(F, c+n)$ is a good pair for every integer n, then c is rational.

Your solution:

You **must** show all your work.

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