The Canadian Mathematical Society La Société mathématique du Canada

## *The Canadian Mathematical Society*

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

# *Sun Life Financial Canadian Open Mathematics Challenge*

Sun Life Financial

### Wednesday, November 25, 2009

**Time:** $2\frac{1}{2}$ hours



**Calculators are NOT permitted.**

Do not open this booklet until instructed to do so.

There are two parts to this paper.

**PART A**

This part of the paper consists of 8 questions, each worth 5 marks. You can earn full value for each question by entering the correct answer(s) in the space provided. If your answer is incorrect, any work that you do will be considered for part marks, **provided that it is done in the space allocated** to that question in your answer booklet.

**PART B**

This part of the paper consists of 4 questions, each worth 10 marks. Finished solutions must be written in the appropriate location in the answer booklet. Rough work should be done separately. If you require extra pages for your finished solutions, paper will be provided by your supervising teacher. Any extra papers should be placed inside your answer booklet. Be sure to write your name and school name on any inserted pages.

Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

**NOTES:**

**At the completion of the contest, insert the information sheet inside the answer booklet.**

**The names of top scoring competitors will be published on the Web sites of the CMS and CEMC.**

## *Sun Life Financial Canadian Open Mathematics Challenge*

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| NOTE: | 1. | Please read the instructions on the front cover of this booklet. |
| | 2. | Write solutions in the answer booklet provided. |
| | 3. | It is expected that all calculations and answers will be expressed as exact numbers such as $4\pi$, $2+\sqrt{7}$, etc., rather than as $12.566\ldots$ or $4.646\ldots$. |
| | 4. | Calculators are **not** allowed. |
| | 5. | Diagrams are not drawn to scale. They are intended as aids only. |

# PART A

1. Determine the value of

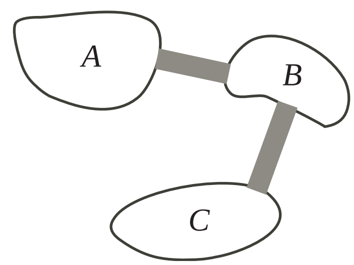
$$-1+2-3+4-5+6-7+8-9+10-11+12-13+14-15+16-17+18\ .$$

2. If $3\times 10^a+5\times 10^b+7\times 10^c=5073$, and $a$, $b$ and $c$ are non-negative integers, what is the value of $a+b+c$?

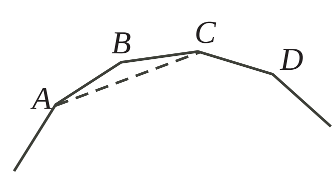
3. Soroosh has 10 coins, each of which is either a quarter (worth 25 cents) or a dime (worth 10 cents). The total value of the dimes is greater than the total value of the quarters. What is the smallest possible number of dimes that he could have?

4. The positive integers 15, 12 and $n$ have the property that the product of any two of them is divisible by the third. Determine the smallest possible value of $n$.

5. In the diagram, there are three islands labelled $A$, $B$ and $C$. Islands $A$ and $B$ are joined by a bridge, as are islands $B$ and $C$. Maya begins her journey on island $A$ and travels between the islands by bridge only. She records the sequence of islands that she visits. She does not necessarily visit all three islands. If Maya makes 20 bridge crossings in total, how many possible sequences of islands $A$, $B$ and $C$ could she travel along?




6. A polygon is called *regular* if all of its sides are equal in length and all of its interior angles are equal in size. In the diagram, a portion of a regular polygon is shown. If $\angle ACD=120^\circ$, how many sides does the polygon have?




7. Determine all angles $\theta$ with $0^\circ\le\theta\le 360^\circ$ such that $\log_2(-3\sin\theta)=2\log_2(\cos\theta)+1$.

8. Determine all triples $(a,b,c)$ of *positive* integers such that $a!=4(b!)+10(c!)$.

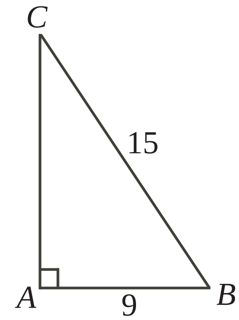
Note: If $n$ is a positive integer, the symbol $n!$ (read as "$n$ factorial") is used to represent the product of the positive integers from 1 to $n$; that is,

$$n!=n(n-1)(n-2)\cdots(3)(2)(1)$$

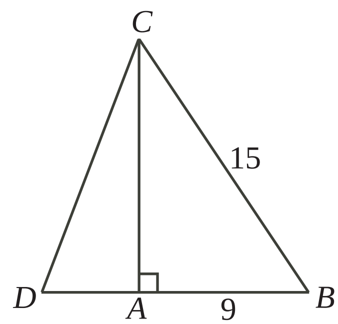
For example, $5!=5(4)(3)(2)(1)$.

## PART B

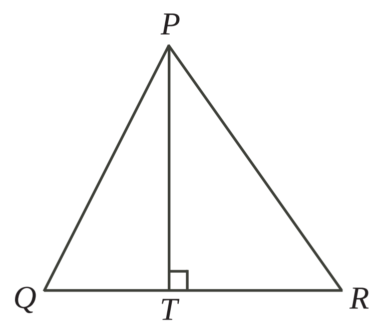
1. (a) In the diagram, $\angle CAB = 90^\circ$, $AB = 9$ and $BC = 15$. Determine the area of $\triangle ABC$.




(b) From part (a), $\triangle ABC$ has side $BA$ extended to $D$. If the area of $\triangle CDB$ is 84, determine the length of $CD$.




(c) In $\triangle PQR$, $PQ = 25$ and $QR = 25$. If the area of $\triangle PQR$ is 300, determine the length of $PR$.




2. Triangle $PQR$ has vertices $P(7, 13)$, $Q(19, 1)$ and $R(1, 1)$. Point $M(4, 7)$ is the midpoint of $PR$; the midpoint of $PQ$ is $N$.

   (a) Determine the equation of the median of the triangle that passes through points $Q$ and $M$.

   (b) Determine the coordinates of $G$, the point of intersection of $RN$ and $QM$.

   (c) Point $F$ is on $PR$ so that $QF$ is perpendicular to $PR$. Point $T$ is on $PQ$ so that $RT$ is perpendicular to $PQ$. Determine the coordinates of $H$, the point of intersection between altitudes $QF$ and $RT$.

   (d) Determine which of $G$ and $H$ is closer to the origin.

3. Suppose that $f$ and $g$ are functions.
   We say that the real number $c$ is a *real fixed point* of $f$ if $f(c) = c$.
   We say that $f$ and $g$ *commute* if $f(g(x)) = g(f(x))$ for all real numbers $x$.

   (a) If $f(x) = x^2 - 2$, determine all real fixed points of $f$.

   (b) If $f(x) = x^2 - 2$, determine all cubic polynomials $g$ that commute with $f$.

   (c) Suppose that $p$ and $q$ are real-valued functions that commute.
   If $2\,[q(p(x))]^4 + 2 = [p(x)]^4 + [p(x)]^3$ for all real numbers $x$, prove that $q$ has no real fixed points.

4. For each positive integer $n$, define $f(n)$ to be the smallest positive integer $s$ for which $1+2+3+\cdots+(s-1)+s$ is divisible by $n$. For example, $f(5) = 4$ because $1+2+3+4$ is divisible by 5 and none of 1, $1+2$, or $1+2+3$ is divisible by 5.

   (a) Determine all positive integers $a$ for which $f(a) = 8$.

   (b) Prove that there are infinitely many odd positive integers $b$ for which

   $$f(b+1) - f(b) > 2009\ .$$

   (c) Determine, with proof, the smallest positive integer $k$ for which the equation $f(c) = f(c+k)$ has an odd positive integer solution for $c$.