The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Canadian Open
Mathematics Challenge

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Solutions

Part A

1. If $x + 2y = 84 = 2x + y$, what is the value of $x + y$?

Solution 1

Since $x + 2y = 84$ and $2x + y = 84$, then adding these two equations together, we obtain $3x + 3y = 168$ or $x + y = 56$.

Solution 2

Since $x + 2y = 84$, then $x = 84 - 2y$.

Substituting into the second equation, we get

$$
\begin{aligned}
2(84 - 2y) + y &= 84 \\
168 - 3y &= 84 \\
84 &= 3y \\
y &= 28
\end{aligned}
$$

Therefore, $x = 84 - 2(28) = 28$ and so $x + y = 56$.

Solution 3

Since $2x + y = 84$, then $y = 84 - 2x$.

Substituting into the first equation, we get

$$
\begin{aligned}
x + 2(84 - 2x) &= 84 \\
168 - 3x &= 84 \\
84 &= 3x \\
x &= 28
\end{aligned}
$$

Therefore, $y = 84 - 2(28) = 28$ and so $x + y = 56$.

Solution 4

SInce these two expressions are identical when x is replaced by y and y is replaced by x, then $x = y$.

Therefore, $3x = 84$ or $x = 28$ and so $y = 28$.

Thus, $x + y = 56$.

ANSWER: 56

2. Let S be the set of all three-digit positive integers whose digits are 3, 5 and 7, with no digit repeated in the same integer. Calculate the remainder when the sum of all of the integers in S is divided by 9.

Solution 1

We can write down the elements of S: 357, 375, 537, 573, 735, 753.

The sum of these elements is $357 + 375 + 537 + 573 + 735 + 753 = 3330$.

Since 3330 is divisible by 9 (because the sum of its digits is divisible by 9), the remainder when we divide by 9 is 0.

Solution 2

There are six numbers formed with the three given numbers.

Two of these numbers have a 3 in the 100s position, two have a 5 in the 100s position, and two have a 7 in the 100s position.

The same can be said about the distribution of numbers in the 10s and units positions.
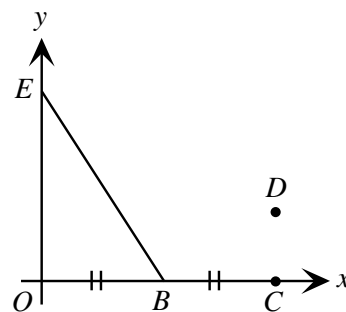
Therefore, the sum of the six numbers is

$$2(3 + 5 + 7)(100) + 2(3 + 5 + 7)(10) + 2(3 + 5 + 7)(1) = 3330$$

The remainder is 0 when 3330 is divided by 9.

ANSWER: 0

3. In the diagram, point E has coordinates $(0, 2)$, and B lies on the positive x-axis so that $BE = \sqrt{7}$. Also, point C lies on the positive x-axis so that $BC = OB$. If point D lies in the first quadrant such that $\angle CBD = 30°$ and $\angle BCD = 90°$, what is the length of ED?

Solution

In order to find the length of ED, we will try to find the coordinates of D. Let the coordinates of B be $(b, 0)$.

Since $BE = \sqrt{7}$ and the coordinates of E are $(0, 2)$, then

$$\begin{aligned}
\sqrt{(b - 0)^2 + (0 - 2)^2} &= \sqrt{7} \\
b^2 + 4 &= 7 \\
b^2 &= 3
\end{aligned}$$

Since the point B lies on the positive x-axis, then $b = \sqrt{3}$ (not $-\sqrt{3}$), so B has coordinates $(\sqrt{3}, 0)$.

(Note that it would have also been possible to find the coordinates of B by using Pythagoras $- OE^2 + OB^2 = EB^2$ so $OB^2 = 7 - 4 = 3$ so $OB = \sqrt{3}$ and so B has coordinates $(\sqrt{3}, 0)$.)

Since $BC = OB$, then C has coordinates $(2\sqrt{3}, 0)$.

Since $\angle BCD = 90°$ and D lies in the first quadrant, then D has coordinates $(2\sqrt{3}, d)$, with $d > 0$.

Since $\triangle DBC$ has $\angle BCD = 90°$ and $\angle CBD = 30°$, then it is a $30° - 60° - 90°$ triangle. Since $CB = \sqrt{3}$ (and CB is opposite the $60°$ angle), then DC (which is opposite the $30°$ angle) has length 1.

Therefore, D has coordinates $(2\sqrt{3}, 1)$.
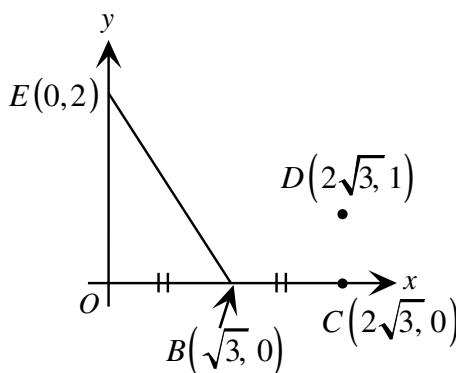
Thus, $ED = \sqrt{(2\sqrt{3} - 0)^2 + (1 - 2)^2} = \sqrt{12 + 1} = \sqrt{13}$.

ANSWER: $\sqrt{13}$

4. A function $f(x)$ has the following properties:

   i) $f(1) = 1$

  ii) $f(2x) = 4f(x) + 6$

 iii) $f(x + 2) = f(x) + 12x + 12$

Calculate $f(6)$.

Solution 1

Using property ii) with $x = 1$,

$$f(2) = 4f(1) + 6 = 4(1) + 6 = 10$$

since $f(1) = 1$ by property i).

Using property ii) with $x = 2$,

$$f(4) = 4f(2) + 6 = 4(10) + 6 = 46$$

Using property iii) with $x = 4$,

$$f(6) = f(4) + 12(4) + 12 = 46 + 48 + 12 = 106$$

Therefore, the value of $f(6)$ is 106.

Solution 2

Using property iii) with $x = 1$,

$$f(3) = f(1) + 12(1) + 12 = 1 + 12 + 12 = 25$$

since $f(1) = 1$ by property i).
Using property ii) with $x = 3$,

$$f(6) = 4f(3) + 6 = 4(25) + 6 = 106$$

Therefore, the value of $f(6)$ is 106.

Solution 3

Working backwards,

$$\begin{aligned}
f(6) &= 4f(3) + 6 \quad \text{(by property ii) with } x = 3) \\
&= 4(f(1) + 12(1) + 12) + 6 \quad \text{(by property iii) with } x = 1) \\
&= 4f(1) + 4(24) + 6 \\
&= 4(1) + 102 \quad \text{(by property i))}
\end{aligned}$$

Therefore, the value of $f(6)$ is 106.

ANSWER: $f(6) = 106$

5. The Rice Tent Company sells tents in two different sizes, large and small. Last year, the Company sold 200 tents, of which one quarter were large. The sale of the large tents produced one third of the company's income. What was the ratio of the price of a large tent to the price of a small tent?

Solution

Since the Rice Tent Company sold 200 tents, of which one quarter were large, then they sold 50 large tents and 150 small tents last year.
Let L be the price of a large tent and S the price of a small tent.
Then their income from large tents was $50L$ and from small tents was $150S$.
Their total income last year was $50L + 150S$.

From the given information,

$$
\begin{aligned}
50L &= \frac{1}{3}(50L + 150S) \\
150L &= 50L + 150S \\
100L &= 150S \\
\frac{L}{S} &= \frac{150}{100} = \frac{3}{2}
\end{aligned}
$$

Therefore, the ratio of the price of a large tent to the price of a small tent was $3 : 2$.

ANSWER: $3 : 2$

6. In the diagram, a square of side length 2 has semicircles drawn on each side. An "elastic band" is stretched tightly around the figure. What is the length of the elastic band in this position?
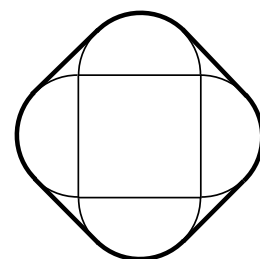

Solution

Label the four vertices of the square as W, X, Y, Z, in clockwise order.

Label the four midpoints of the sides of the square (that is, the centres of the four semicircles) as M, N, O, P, in clockwise order, starting with M being the midpoint of WX.

In each semicircle, join the centre to the two points on that semicircle where the band just starts (or stops) to contact the circle. Label these eight points as A, B, C, D, E, F, G, and H in clockwise order, starting with A and B on the semicircle with centre M.
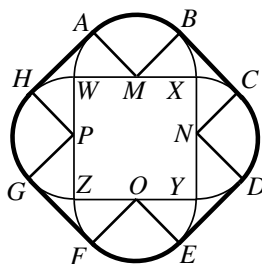
By symmetry, the four straight parts of the band will be equal in length (that is, $BC = DE = FG = HA$) and the four arc segments of the band will be equal in length (that is, $AB = CD = EF = GH$).

Therefore, the total length of the band is $4(\text{Length of arc } AB) + 4(\text{Length of } BC)$.

Now, BC will actually be tangent to the two semicircles (with centres M and N) where it initially just touches them.

Thus, MB and NC are both perpendicular to BC.

Since $MB = NC = 1$ (because they are radii of the semicircles and each semicircle has diameter 2), then $MBCN$ must actually be rectangle, so BC is equal and parallel to MN.

Since M and N are the midpoints of the sides of the square of side length 2, then $MY = YN = 1$, so $MN = \sqrt{2}$, so $BC = \sqrt{2}$.

Next, we determine the length of AB. Previously, we saw that $MBCN$ is a rectangle, so BC was parallel to MN. Similarly, HA is parallel to PM.

But PM is perpendicular to MN, so HA is perpendicular to BC.

Therefore, $\angle AMB = 90°$, ie. AB is one-quarter of the circumference of a circle with radius 1 or $\frac{1}{4}(2\pi(1)) = \frac{1}{2}\pi$.

Therefore, the total length of the band is $4(\frac{1}{2}\pi) + 4(\sqrt{2}) = 2\pi + 4\sqrt{2}$.

ANSWER: $2\pi + 4\sqrt{2}$

7. Let a and b be real numbers, with $a > 1$ and $b > 0$.
   If $ab = a^b$ and $\dfrac{a}{b} = a^{3b}$, determine the value of a.

Solution 1
Since $ab = a^b$ and $\dfrac{a}{b} = a^{3b}$, then multiplying these two equations together, we get

$$a^2 = a^b \cdot a^{3b} = a^{4b}.$$

Dividing both sides by a^2 (since $a \neq 0$), we get $a^{4b-2} = 1$.
Since $a > 1$, then $4b - 2 = 0$ or $b = \frac{1}{2}$.
Substituting back into the first equation, we get $\frac{1}{2}a = a^{1/2} = \sqrt{a}$ or $a = 2\sqrt{a}$.
Squaring both sides gives $a^2 = 4a$ or $a^2 - 4a = 0$ or $a(a - 4) = 0$.
Since $a > 1$, then $a = 4$.

Solution 2
Since $ab = a^b$, then, dividing both sides by a which is not equal to 0, we get $b = a^{b-1}$.
Since $\dfrac{a}{b} = a^{3b}$, then $a = ba^{3b} = a^{b-1}a^{3b} = a^{4b-1}$.
Comparing exponents, we get $1 = 4b - 1$ or $b = \frac{1}{2}$.
Substituting $b = \frac{1}{2}$ into $ab = a^b$, we have $\frac{1}{2}a = a^{1/2}$ or $a^{1/2} = 2$ or $a = 4$. So $a = 4$.

Solution 3
Since $a > 1$ and $b > 0$, we can take logarithms of both sides of both equations.
In the first equation, using log rules on $\log(ab) = \log\left(a^b\right)$ gives $\log(a) + \log(b) = b\log(a)$.
In the first equation, using log rules on $\log\left(\frac{a}{b}\right) = \log\left(a^{3b}\right)$ gives $\log(a) - \log(b) = 3b\log(a)$.
Adding these two new equations gives $2\log(a) = 4b\log(a)$ or $(4b - 2)\log(a) = 0$.
Since $a > 1$, then $\log(a) > 0$, so we must have $4b - 2 = 0$ or $b = \frac{1}{2}$.

Substituting this back into the first log equation gives $\log(a) + \log\left(\frac{1}{2}\right) = \frac{1}{2}\log(a)$ or $\frac{1}{2}\log(a) = -\log\left(\frac{1}{2}\right) = \log(2)$ or $\log(a) = 2\log(2) = \log(4)$, so $a = 4$.

ANSWER: 4

8. A rectangular sheet of paper, $ABCD$, has $AD = 1$ and $AB = r$, where $1 < r < 2$. The paper is folded along a line through A so that the edge AD falls onto the edge AB. Without unfolding, the paper is folded again along a line through B so that the edge CB also lies on AB. The result is a triangular piece of paper. A region of this triangle is four sheets thick. In terms of r, what is the area of this region?
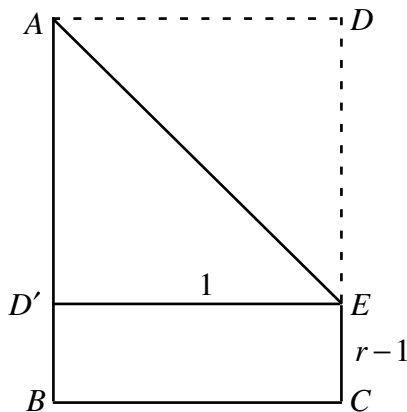
Solution

Start with the rectangular sheet of paper, $ABCD$, with A in the top left and B in the bottom left.

Fold AD across so that AD lies along AB. Let D' be the point were D touches AB and let E be the point on DC where the fold hits DC.

Since AD' is the old AD, then $AD' = 1$.

Since $D'E$ is perpendicular to $D'A$ (since DC was perpendicular to AD) then $D'E$ is parallel to BC, so $D'E = 1$ as well.
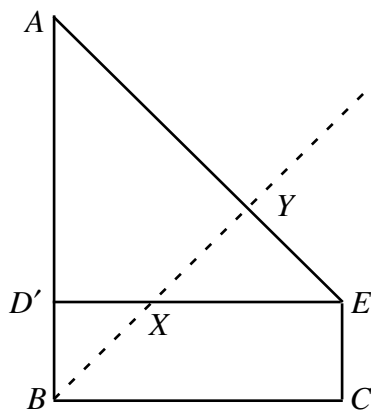

We can also conclude that $D'B = EC = r - 1$, since AB has length r.

Next, we fold the paper so that BC lies along AB. Unfold this paper and lay it flat so that we can see the crease.

Since BC is folded onto AB, then the crease bisects $\angle ABC$, that is the crease makes an angle of $45°$ with both AB and BC.

Suppose that the crease crosses $D'E$ at X and AE at Y.

Now when the paper had been folded the second time (before we unfolded it!), the only way to obtain a region four sheets thick was to fold a region two sheets thick on top of a region which is also two sheets thick.

Since $\triangle XYE$ is the only part of the paper "below" the second crease which is two sheets thick, and, when the second fold is made, it lies entirely over another region which is two sheets thick, then the desired area is the area of $\triangle XYE$.

Since $\angle ABX = 45°$, then $\triangle BD'X$ is isosceles and right-angled, so $D'X = D'B = r - 1$.
Thus, $EX = 1 - D'X = 1 - (r - 1) = 2 - r$.
Since $\angle D'XB = 45°$, then $\angle YXE = 45°$. Also, since $\triangle AD'E$ is right isosceles, then $\angle YEX = 45°$, so $\triangle XYE$ is isosceles and right-angled.
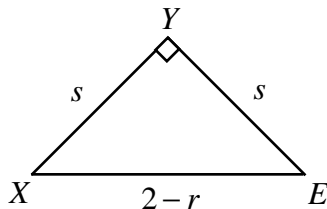
Suppose $XY = s$. Then $\sqrt{2}s = XE = 2 - r$ or $2s^2 = (2 - r)^2$.
The area of $\triangle XYE$ is $\frac{1}{2}s^2$ or $\frac{1}{4}(2 - r)^2$.
Therefore, the area of the desired region is $\frac{1}{4}(2 - r)^2$.

ANSWER: $\frac{1}{4}(2 - r)^2$

Part B

1. The points $A(-8, 6)$ and $B(-6, -8)$ lie on the circle $x^2 + y^2 = 100$.

   (a) Determine the equation of the line through A and B.

   Solution
   First, we determine the slope of the line segment AB. The slope is $\dfrac{6 - (-8)}{-8 - (-6)} = -7$.
   We could now proceed to find the equation of the line in several different ways.
   Using the point-slope form, we obtain $y - 6 = -7(x - (-8))$ or $y = -7x - 50$.

   (b) Determine the equation of the perpendicular bisector of AB.

   Solution
   Since the slope of AB is -7, then the slope of the perpendicular bisector of AB is $\frac{1}{7}$.
   Also, the perpendicular bisector passes through the midpoint of AB, which is

   $$\left(\frac{1}{2}((-8) + (-6)), \frac{1}{2}(6 + (-8)) \right) = (-7, -1).$$

   Therefore, the equation of the perpendicular bisector is $y - (-1) = \frac{1}{7}(x - (-7))$ or $y = \frac{1}{7}x$.

   (c) The perpendicular bisector of AB cuts the circle at two points, P in the first quadrant and Q in the third quadrant. Determine the coordinates of P and Q.

   Solution 1
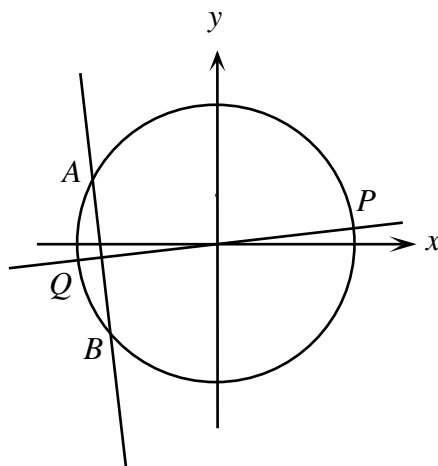
   We want to find the points of intersection of the circle $x^2 + y^2 = 100$ and the line $y = \frac{1}{7}x$.
   From the equation of the line, $x = 7y$. Substituting this into the equation of the circle we

obtain

$$
\begin{aligned}
(7y)^2 + y^2 &= 100 \\
49y^2 + y^2 &= 100 \\
50y^2 &= 100 \\
y^2 &= 2 \\
y &= \pm\sqrt{2}
\end{aligned}
$$

Since $x = 7y$, then if $y = \sqrt{2}$, then $x = 7\sqrt{2}$, and if $y = -\sqrt{2}$, then $x = -7\sqrt{2}$.
Since P is in the first quadrant, then P has coordinates $(7\sqrt{2}, \sqrt{2})$.
Since Q is in the third quadrant, then Q has coordinates $(-7\sqrt{2}, -\sqrt{2})$.

Solution 2
We want to find the points of intersection of the circle $x^2 + y^2 = 100$ and the line $y = \frac{1}{7}x$.
Substituting $y = \frac{1}{7}x$ into the equation of the circle we obtain

$$
\begin{aligned}
x^2 + \left(\frac{1}{7}x\right)^2 &= 100 \\
x^2 + \frac{1}{49}x^2 &= 100 \\
\frac{50}{49}x^2 &= 100 \\
x^2 &= 98 \\
x &= \pm\sqrt{98} = \pm 7\sqrt{2}
\end{aligned}
$$

Since $y = \frac{1}{7}x$, then if $x = 7\sqrt{2}$, then $y = \sqrt{2}$, and if $x = -7\sqrt{2}$, then $y = -\sqrt{2}$.
Since P is in the first quadrant, then P has coordinates $(7\sqrt{2}, \sqrt{2})$.
Since Q is in the third quadrant, then Q has coordinates $(-7\sqrt{2}, -\sqrt{2})$.

(d) What is the length of PQ? Justify your answer.

Solution 1
The points P and Q are joined by the line $y = \frac{1}{7}x$, which passes through the origin.
Since the origin is the centre of the circle, then PQ must be a diameter of the circle.
Since the circle has equation $x^2 + y^2 = 100 = 10^2$, then its radius is 10, so its diameter is 20.
Therefore, $PQ = 20$.

Solution 2
Since we know that $P(7\sqrt{2}, \sqrt{2})$ and $Q(-7\sqrt{2}, -\sqrt{2})$, then we can determine the distance

PQ by direct calculation:

$$\begin{aligned} PQ &= \sqrt{\left(7\sqrt{2} - \left(-7\sqrt{2}\right)\right)^2 + \left(\sqrt{2} - \left(-\sqrt{2}\right)\right)^2} \\ &= \sqrt{\left(14\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} \\ &= \sqrt{\left(\sqrt{2}\right)^2 [14^2 + 2^2]} \\ &= \sqrt{2[196 + 4]} \\ &= \sqrt{400} \\ &= 20 \end{aligned}$$

Therefore, the length of PQ is 20.

2. (a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.

Solution
Factoring the given equation $x^2 - 4x - 12 = 0$, we obtain $(x - 6)(x + 2) = 0$.
Therefore, the two solutions are $x = 6$ and $x = -2$.

(b) Determine the *one* value of x such that $x - \sqrt{4x + 12} = 0$. Justify your answer.

Solution
We first eliminate the square root by isolating it on one side and squaring:

$$\begin{aligned} x - \sqrt{4x + 12} &= 0 \\ x &= \sqrt{4x + 12} \\ x^2 &= 4x + 12 \\ x^2 - 4x - 12 &= 0 \\ (x - 6)(x + 2) &= 0 \end{aligned}$$

Therefore, the two possible solutions are $x = 6$ and $x = -2$. (Since we have squared both sides, it is possible that we have introduced an extraneous root, so we should verify both of these.)
If $x = 6$, then $6 - \sqrt{4(6) + 12} = 6 - \sqrt{36} = 0$.
If $x = -2$, then $(-2) - \sqrt{4(-2) + 12} = -2 - \sqrt{4} = -4 \neq 0$.
Therefore, the one value of x that solves the equation is $x = 6$.

(c) Determine all real values of c such that

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

has precisely two distinct real solutions for x.

Solution

We start by attempting to solve this equation and then seeing what conditions on c arise.
Since $8x^2 - 32x - 8c = 8(x^2 - 4x - c)$, we let $T = x^2 - 4x - c$.
Then the equation is

$$
\begin{aligned}
T - \sqrt{8T} &= 0 \quad (*) \\
T &= \sqrt{8T} \\
T^2 &= 8T \\
T(T - 8) &= 0
\end{aligned}
$$

Therefore, $T = 0$ or $T = 8$. We can check that neither root is extraneous, so $x^2 - x - c = 0$
or $x^2 - 4x - c = 8$.
Let's look at these last two equations.
First, we look at $x^2 - 4x - c = 0$. The discriminant of this quadratic equation is $(-4)^2 - 4(-c) = 16 + 4c$. Therefore, this equation has

- zero solutions if $16 + 4c < 0$, so $c < -4$,
- exactly one solution if $16 + 4c = 0$, so $c = -4$, and
- two distinct solutions if $16 + 4c > 0$, so $c > -4$.

Next, we look at $x^2 - 4x - c = 8$ or $x^2 - 4x - (c + 8) = 0$. The discriminant of this
quadratic equation is $(-4)^2 - 4(-(c + 8)) = 48 + 4c$. Therefore, this equation has

- zero solutions if $48 + 4c < 0$, so $c < -12$,
- exactly one solution if $48 + 4c = 0$, so $c = -12$, and
- two distinct solutions if $48 + 4c > 0$, so $c > -12$.

We see also that any value of x that satisfies one of these two equations cannot satisfy the
other, since we cannot have both $x^2 - 4x - c = 0$ and $x^2 - 4x - c = 8$. (In other words,
no roots overlap between these two equations.)
We make a table to combine our observations:

	$c < -12$	$c = -12$	$-12 < c < -4$	$c = -4$	$c > -4$
$x^2 - 4x - c = 0$	0 solutions	0 solutions	0 solutions	1 solution	2 solutions
$x^2 - 4x - c = 8$	0 solutions	1 solution	2 solutions	2 solutions	2 solutions
Total solutions	0 solutions	1 solution	2 solutions	3 solutions	4 solutions

Therefore, for there to be exactly two distinct solutions, we must have $-12 < c < -4$.

3. A map shows all Beryl's Llamaburgers restaurant locations in North America. On this map, a line segment is drawn from each restaurant to the restaurant that is closest to it. Every restaurant has a unique closest neighbour. (Note that if A and B are two of the restaurants, then A may be the closest to B without B being closest to A.)

   (a) Prove that no three line segments on the map can form a triangle.

   Solution 1

   We start by assuming that three line segments on the map do form a triangle, and show that this is in fact impossible.

   Notice that if restaurants X and Y are joined by a line segment, then either X is the closest restaurant to Y or Y is the closest restaurant to X (or both).

   Assume that A, B and C are the three points on the map connect by segments.
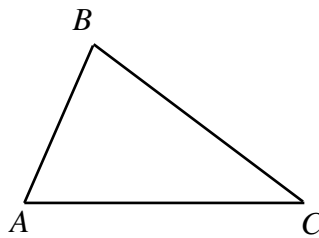
   To begin, we focus on the segment joining A to B. Let's assume that A is the closest restaurant to B. (It doesn't matter which direction we assume here.) This means that C is not the closest restaurant to B, so $BA < BC$.

   But B and C are connected and C is not the closest restaurant to B. Therefore, B is the closest restaurant to C, which means $CB < CA$.

   But C and A are also connected and A is not the closest restaurant to C. Therefore, C is the closest restaurant to A, which means $AC < AB$.

   But this means that $BA < BC$, $BC < AC$ and $AC < BA$. This cannot be the case. Therefore, it is impossible for three line segments to form a triangle.

   Solution 2

   We prove this by showing that constructing a triangle is impossible.

   We start by considering two locations A and B and the line segment AB.

   Since A and B are connected, we can assume without loss of generality that A is closest to B. (The case B closest to A involves interchanging A and B, and the case of A and B closest to each other is included in the case of A closest to B.)

   If A is closest to B and we add a new location C which is connected to B, then B must be closest to C (since C can't also be closest to B along with A).

If we join C to A, then either C is closest to A or A is closest to C.

But A can't be closest to C since B is closest to C.

Therefore, we must have C closest to A.

But then AC is shorter than AB, along with AB being shorter than BC (since A is closest to B), which means that AC is shorter than BC or A is closer to C than B is, which isn't true. This is a contradiction.

Therefore, we can't construct a triangle.

(b) Prove that no restaurant can be connected to more than five other restaurants.

Solution

We start by assuming that one restaurant can be connected to six others and show that this is impossible. From this we can conclude that no restaurant can be connected to more than five other restaurants (for if it could be joined to 8 others, say, then we could consider six of them only and reach a contradiction).

Assume that restaurant A can be connected to restaurants B, C, D, E, F, and H, where these restaurants are listed in clockwise order of their line segments joining to A.
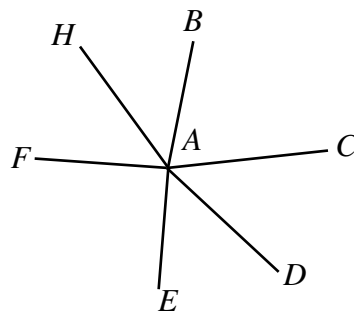
Consider restaurants A, B and C.

We know that B and C are both connected to A and both cannot be the closest neighbour to A. Thus, A must be the closest neighbour to one of these, say B. (It doesn't matter which we choose).

Since A is the closest restaurant to B, then $BA < BC$.

Now consider the line joining C to A.

If C is the closest neighbour to A, then $AC < AB$, so $AC < AB < BC$.

If A is the closest neighbour to C, then $CA < CB$ so $CA < CB$ and $BA < CB$.

In either case, BC is the (strictly) longest side in $\triangle ABC$, and so must be opposite the (strictly) largest angle.

Since the angles in a triangle add to $180°$, then if there is a largest angle, then this angle must be larger than $60°$. Therefore, from the above reasoning, $\angle BAC > 60°$.

But we can reapply this reasoning to conclude that $\angle CAD$, $\angle DAE$, $\angle EAF$, $\angle FAH$, and $\angle HAB$ are each greater than $60°$. But the sum of these six angles is $360°$, since they will form a full circle around A, and six angles, each greater than $60°$, cannot add to $360°$. So we have a contradiction.

Therefore, it is impossible for a restaurant to be connected to more than five other restaurants.

4. In a *sumac sequence*, t_1, t_2, t_3, ..., t_m, each term is an integer greater than or equal to 0. Also, each term, starting with the third, is the difference of the preceding two terms (that is, $t_{n+2} = t_n - t_{n+1}$ for $n \geq 1$). The sequence terminates at t_m if $t_{m-1} - t_m < 0$. For example, 120, 71, 49, 22, 27 is a sumac sequence of length 5.

(a) Find the positive integer B so that the sumac sequence 150, B, ... has the maximum possible number of terms.

Solution

Suppose that we have a sumac sequence with $t_1 = 150$ and $t_2 = B$. Let's write out the next several terms (assuming that they exist) in terms of B:

$$t_3 = 150 - B \qquad t_4 = 2B - 150 \qquad t_5 = 300 - 3B \qquad t_6 = 5B - 450$$
$$t_7 = 750 - 8B \qquad t_8 = 13B - 1200 \qquad t_9 = 1950 - 21B \qquad t_{10} = 34B - 3150$$

In order to maximize the length of this sumac sequence, we would like to choose B so that as many terms as possible starting from t_1 are non-negative. (When we reach the first negative "term", we know that the sequence terminated at the previous term.)

For $t_2 \geq 0$, $B \geq 0$.
For $t_3 \geq 0$, $150 - B \geq 0$ or $B \leq 150$.
For $t_4 \geq 0$, $2B - 150 \geq 0$ or $B \geq 75$.
For $t_5 \geq 0$, $300 - 3B \geq 0$ or $B \leq 100$.
For $t_6 \geq 0$, $5B - 450 \geq 0$ or $B \geq 90$.
For $t_7 \geq 0$, $750 - 8B \geq 0$ or $B \leq \frac{750}{8} = 93\frac{6}{8}$.
For $t_8 \geq 0$, $13B - 1200 \geq 0$ or $B \geq \frac{1200}{13} = 92\frac{4}{13}$.
For $t_9 \geq 0$, $1950 - 21B \geq 0$ or $B \leq \frac{1950}{21} = 92\frac{18}{21}$.

Therefore, since B must be a positive integer, if we choose $B = 93$, then we can ensure that each of t_1 through t_8 are non-negative. This will maximize the number of terms starting from the beginning, since B must satisfy $92\frac{4}{13} \leq B \leq 93\frac{6}{8}$ in order for at least the first eight terms to be non-negative. (Note that t_9 will in fact be negative when $B = 93$.)

When we set $B = 93$, we obtain the sumac sequence 150, 93, 57, 36, 21, 15, 6, 9.

(b) Let m be a positive integer with $m \geq 5$. Determine the number of sumac sequences of length m with $t_m \leq 2000$ and with no term divisible by 5.

Solution

We begin our solution by making some observations about sumac sequences.

- A sumac sequence is completely determined by its first two terms. This is true since the first two terms give us the third, the second and third give us the fourth, and so on. The sequence will terminate when the "next term" would be negative.

- In a sumac sequence, since for every (valid) n we have $t_{n+2} = t_n - t_{n+1}$, then $t_n = t_{n+1} + t_{n+2}$. This means that we can "reverse engineer" a sumac sequence – if we know terms $(n+1)$ and $(n+2)$, then we can determine term n. Thus, if we know the final two terms in a sumac sequence, then we can determine all of the previous terms.

- From the first observation, the first two terms of a sumac sequence completely determines the sequence. Is the same true of the last two terms? No. When we start looking at a sumac sequence from the back, every new term as we proceed towards the front will always be non-negative (since we are adding non-negative terms). Thus, there is no "stopping condition" as there is when we work forwards. (For example, 3, 1, 2 is a sumac sequence ending with 1, 2, as is 4, 3, 1, 2.)

- However, if we know the final two terms and the length of the sequence, this completely determines the sumac sequence (and we will always be able to find such a sequence).

Now we proceed. Let m be a fixed positive integer with $m \geq 5$.

Suppose that $t_1, t_2, \ldots, t_{m-1}, t_m$ is a sumac sequence of length m.

Because we are given a condition on the final term of the sequence, we will examine the sequence from the back.

Let $x = t_m$ and $y = t_{m-1}$. Note that x, y and m determine the sequence.

Since x and y are the last two terms in the sumac sequence, then $t_{m-1} - t_m = y - x < 0$ or $x > y$.

Since we have m fixed, we would like to determine how many sumac sequences we can form with $t_m = x \leq 2000$, $t_{m-1} = y < x$ and no term divisible by 5.

Let's write out the last five terms of the sequence (in reverse order): x, y, $x + y$, $x + 2y$, $2x + 3y$. (Since $m \geq 5$, we know that there are at least five terms in the sequence.)

Since we want no term divisible by 5, let us consider x and y modulo 5 to see what happens. (There are 25 possible pairs for (x, y) modulo 5.)

Since no term is divisible by 5, then we don't want $x \equiv 0 \pmod 5$ or $y \equiv 0 \pmod 5$. This cuts us down to 16 possibilities for (x, y).

We make a table of these possibilities to determine which pairs can be eliminated simply by looking at the last five terms. (All entries in the table are modulo 5. In any given row, we stop if we reach a 0, since this possibility can then be discarded.)

x	y	$x + y$	$x + 2y$	$2x + 3y$
1	1	2	3	0
1	2	3	0	
1	3	4	2	1
1	4	0		
2	1	3	4	2
2	2	4	1	0
2	3	0		
2	4	1	0	
3	1	4	0	
3	2	0		
3	3	1	4	0
3	4	2	1	3
4	1	0		
4	2	1	3	4
4	3	2	0	
4	4	3	2	0

So the only possible pairs for (x, y) modulo 5 are $(1, 3)$, $(2, 1)$, $(3, 4)$ and $(4, 2)$.

If we start with $(x, y) = (1, 3)$ modulo 5, then the terms in the sequence modulo 5 are 1, 3, 4, 2, 1, 3, 4, 2, 1, ..., ie. the terms cycle modulo 5 with no terms divisible by 5.

This similar cycling will happen with each of the other 3 pairs, so each of these 4 pairs give no terms divisible by 5.

So for each of these pairs, we need to determine the number of pairs of non-negative integers (x, y) with $x \leq 2000$, $y < x$ and congruent to the appropriate things modulo 5. Each such pair will give a sumac sequence of length $m \geq 5$ with no term divisible by 5. (Since the divisibility of the terms is independent of length, this also means that the number of such sequences will be independent of m!)

Case 1: (x, y) congruent to $(1, 3)$ modulo 5

Here x can take the values 1996, 1991, ..., 6, 1.

If $x = 1996$, y can be 1993, 1988, ..., 8, 3. (399 possibilities)

If $x = 1991$, y can be 1988, 1983, ..., 8, 3. (398 possibilities)

This pattern continues, with one fewer possibility each time x decreases by 5, until we reach $x = 6$, where $y = 3$ is the only possibility.

Thus, there are $399 + 398 + \cdots + 2 + 1$ possibilities for this case.

Case 2: (x, y) congruent to $(2, 1)$ modulo 5

Here x can take the values 1997, 1992, ..., 7, 2.

If $x = 1997$, y can be 1996, 1991, ..., 6, 1. (400 possibilities)

If $x = 1992$, y can be 1991, 1986, ..., 6, 1. (399 possibilities)

This pattern continues, with one fewer possibility each time x decreases by 5, until we reach $x = 2$, where $y = 1$ is the only possibility.

Thus, there are $400 + 399 + \cdots + 2 + 1$ possibilities for this case.

Case 3: (x, y) congruent to $(3, 4)$ modulo 5

Here x can take the values 1998, 1993, ..., 8, 3.

If $x = 1998$, y can be 1994, 1989, ..., 9, 4. (399 possibilities)

If $x = 1993$, y can be 1989, 1984, ..., 9, 4. (398 possibilities)

This pattern continues, with one fewer possibility each time x decreases by 5, until we reach $x = 8$, where $y = 4$ is the only possibility.

Thus, there are $399 + 398 + \cdots + 2 + 1$ possibilities for this case.

Case 4: (x, y) congruent to $(4, 2)$ modulo 5

Here x can take the values 1999, 1994, ..., 9, 4.

If $x = 1999$, y can be 1997, 1992, ..., 7, 2. (400 possibilities)

If $x = 1994$, y can be 1992, 1987, ..., 7, 2. (399 possibilities)

This pattern continues, with one fewer possibility each time x decreases by 5, until we reach $x = 4$, where $y = 2$ is the only possibility.

Thus, there are $400 + 399 + \cdots + 2 + 1$ possibilities for this case.

Therefore, overall there are

$$2(399+398+\cdots+2+1)+2(400+399+\cdots+2+1) = 399(400)+400(401) = 400(800) = 320\,000$$

possibilities. Therefore, there are exactly $320\,000$ sumac sequences of length m with no term divisible by 5 and with $t_m \leq 2000$.