The Canadian Mathematical Society

in collaboration with

The CENTRE for Education
in MATHEMATICS and COMPUTING

The Canadian Open Mathematics Challenge

Wednesday, November 26, 2003

Solutions

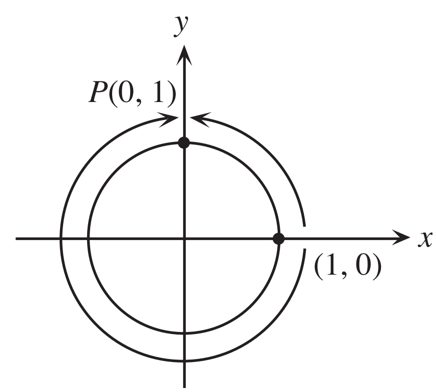
Part A

1. Let Gareth's present age, in years, be G.
   Then Jeff's age is $G - 1$, and Ina's age is $G + 2$.
   Since the sum of their three ages is 118, then
   $$(G-1) + G + (G+2) = 118$$
   $$G = 39$$
   Therefore, Gareth's age is 39.

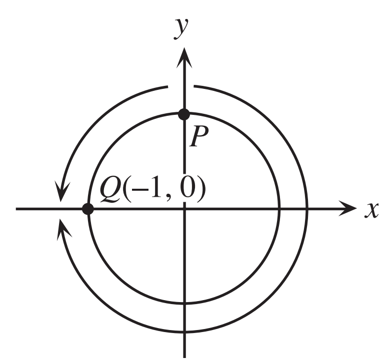
2. When the point $(4,-2)$ is reflected in the x-axis, its image is $(4,2)$.
   When the point $(4,2)$ is reflected in the line $y = x$, its image is $(2,4)$.
   Therefore, the coordinates of the final point are $(2,4)$.

3. The particle which moves clockwise is moving three times as fast as the particle moving counterclockwise. Therefore, the particle moving clockwise moves three times as far as the particle moving counterclockwise in the same amount of time.

   This tells us that in the time that the clockwise particle travels $\frac{3}{4}$ of the way around the circle, the counterclockwise particle will travel $\frac{1}{4}$ of the way around the circle, and so the two particles will meet at $P(0,1)$.



   Using the same reasoning, the particles will meet at $Q(-1,0)$ when they meet the second time.



4. *Solution 1*

In choosing a pair of numbers from the five given numbers, there are 10 different ways of choosing these numbers. These pairs are $\{(0,1),(0,2),(0,3),(0,4),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$. The only pairs in which the sum is greater than the product are those containing a 0 or a 1. Since there are 7 of these, the required probability is $\frac{7}{10}$.

Solution 2

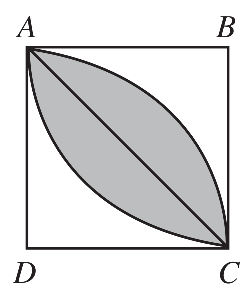
The most straightforward way to approach this problem is to make a chart:

Numbers chosen	Sum	Product
0,1	1	0
0,2	2	0
0,3	3	0
0,4	4	0
1,2	3	2
1,3	4	3
1,4	5	4
2,3	5	6
2,4	6	8
3,4	7	12

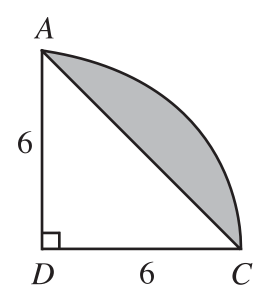
So there are 10 possible ways that two different numbers can be chosen, and for 7 of these possibilities, the sum of the two numbers is greater than the product.
Therefore, the probability is $\frac{7}{10}$.

5. Join A to C.

This line divides the shaded region into two identical pieces.



Consider the shaded region above AC.
This piece of the region is formed by taking the sector DAC of the circle, centre D and radius 6, and then removing ΔADC.



Since $\angle ADC = 90^\circ$, then the sector is one quarter of the whole circle, and has area $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi\left(6^2\right) = 9\pi$.

Also, ΔADC is right-angled with base DC of length 6 and height DA of length 6, and so has area $\frac{1}{2}bh = \frac{1}{2}(6)(6) = 18$.

Therefore, the area of the region above the line is $9\pi - 18$, and so the area of the entire shaded region is $2(9\pi - 18) = 18\pi - 36$ square units.

6. If $x < 0$, then $\frac{3}{x} < 0$, so $\left\lfloor \frac{3}{x} \right\rfloor \leq \frac{3}{x} < 0$. Similarly, $\left\lfloor \frac{4}{x} \right\rfloor < 0$, so we cannot possibly have $\left\lfloor \frac{3}{x} \right\rfloor + \left\lfloor \frac{4}{x} \right\rfloor = 5$. Therefore, $x > 0$.

When $x > 0$, we have $\frac{3}{x} < \frac{4}{x}$, so $\left\lfloor \frac{3}{x} \right\rfloor \leq \left\lfloor \frac{4}{x} \right\rfloor$.

Since each of $\left\lfloor \frac{3}{x} \right\rfloor$ and $\left\lfloor \frac{4}{x} \right\rfloor$ is an integer, then there are three possibilities:

i) $\left\lfloor \frac{3}{x} \right\rfloor = 0$ and $\left\lfloor \frac{4}{x} \right\rfloor = 5$

ii) $\left\lfloor \frac{3}{x} \right\rfloor = 1$ and $\left\lfloor \frac{4}{x} \right\rfloor = 4$

iii) $\left\lfloor \frac{3}{x} \right\rfloor = 2$ and $\left\lfloor \frac{4}{x} \right\rfloor = 3$

If $\left\lfloor \frac{3}{x} \right\rfloor = 0$, then $0 \leq \frac{3}{x} < 1$ or $x > 3$. If $\left\lfloor \frac{4}{x} \right\rfloor = 5$, then $5 \leq \frac{4}{x} < 6$ or $\frac{2}{3} < x \leq \frac{4}{5}$. These intervals do not overlap, so there are no solutions in this case.

If $\left\lfloor \frac{3}{x} \right\rfloor = 1$, then $1 \leq \frac{3}{x} < 2$ or $\frac{3}{2} < x \leq 3$. If $\left\lfloor \frac{4}{x} \right\rfloor = 4$, then $\frac{4}{5} < x \leq 1$. These intervals do not overlap, so there are no solutions in this case.

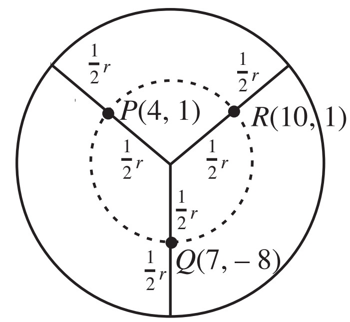
If $\left\lfloor \frac{3}{x} \right\rfloor = 2$, then $2 \leq \frac{3}{x} < 3$ or $1 < x \leq \frac{3}{2}$. If $\left\lfloor \frac{4}{x} \right\rfloor = 3$, then $1 < x \leq \frac{4}{3}$. In this case, the intervals do overlap. When we combine these intervals, we see that if $1 < x \leq \frac{4}{3}$, then $\left\lfloor \frac{3}{x} \right\rfloor + \left\lfloor \frac{4}{x} \right\rfloor = 5$.

Therefore, the range of values is $1 < x \leq \frac{4}{3}$.

7. *Solution 1*

Let the radius of circle C be r.

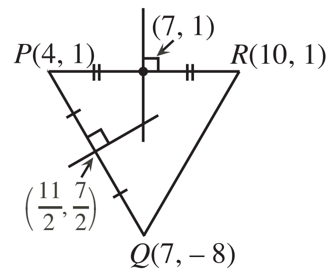
Since P, Q and R are given as midpoints of the radius of the large circle, they themselves lie on a circle with the same centre as the given circle, but with half its radius.



Method 1 – Perpendicular bisectors

To find the centre of the circle passing through P, Q and R, we must find the intersection of the perpendicular bisectors of the sides of the triangle formed by the three points.

Consider first side PR. Since PR is a line segment parallel to the x-axis, its perpendicular bisector has equation $x = 7$.

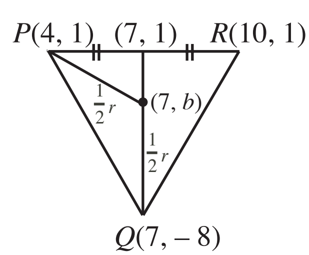


Consider next side PQ. Since P has coordinates $(4,1)$ and Q has coordinates $(7,-8)$, then PQ has slope –3 and has midpoint $\left(\frac{11}{2},-\frac{7}{2}\right)$. Therefore, the perpendicular bisector of PQ has slope $\frac{1}{3}$ and has equation $y+\frac{7}{2}=\frac{1}{3}\left(x-\frac{11}{2}\right)$.

Therefore, at the intersection of these two perpendicular bisectors, $y+\frac{7}{2}=\frac{1}{3}\left(7-\frac{11}{2}\right)$ or $y=-3$. Thus the centre of the circle is the point $(7,-3)$, and since $(7,-8)$ lies on the circle, the radius of the small circle is 5. Therefore, the radius of circle C is 10.

Method 2 – Geometric

As in Method 1, we proceed by trying to find the centre of the circle. Also, we again know that the centre is the intersection of the perpendicular bisectors of the sides of the ΔPQR. One perpendicular bisector is very easy to find – that of PR, which has equation $x = 7$ as we found above. This tells us that the centre lies on the line $x = 7$. Thus, the centre of the circle can be represented by $O(7,b)$.



Since radii of a circle are equal,

$$OP^2 = OQ^2$$
$$(7-4)^2 + (b-1)^2 = (7-7)^2 + (b+8)^2$$
$$9 + b^2 - 2b + 1 = b^2 + 16b + 64$$
$$b = -3$$

The radius of the circle P, Q and R is $\sqrt{3^2 + (-4)^2} = 5$ and the radius of the larger circle is 10.

8. The first thing that we must notice in this problem is because we are looking for positive integers k, l and m such that

$$\frac{4k}{5} + \frac{5l}{6} + \frac{6m}{7} = 82$$

then k must be divisible by 5, l must be divisible by 6, and m must be divisible by 7.

So we make the substitution $k = 5K$, $l = 6L$ and $m = 7M$, where K, L and M are positive integers.

Therefore, we obtain, by substitution

$$5K + 6L + 7M = 97$$
$$4K + 5L + 6M = 82$$

Subtracting the second equation from the first, we get

$$K + L + M = 15$$
$$4K + 5L + 6M = 82$$

Subtracting six times the first equation from the second equation, we get

$$K + L + M = 15$$
$$-2K - L = -8$$

or

$$K + L + M = 15$$
$$2K + L = 8$$

Since K, L and M are all positive integers, we obtain from the second equation the following possibilities for K and L which give us M from the first equation, and thus k, l and m:

K	L	M	k	l	m
1	6	8	5	36	56
2	4	9	10	24	63
3	2	10	15	12	70

Therefore, there are three triples (k, l, m) of positive integers which are solutions to the system of equations.

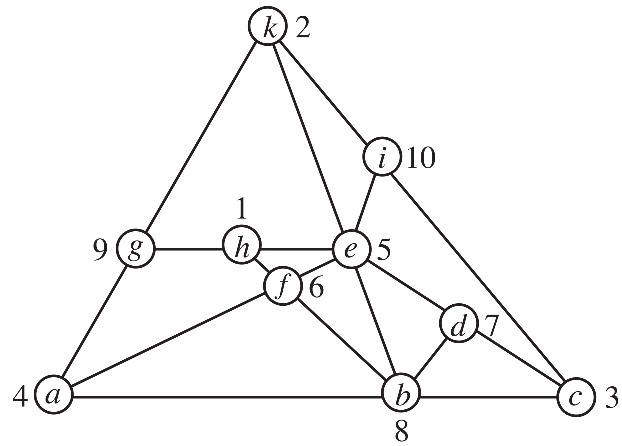
Part B

1. (a) We will proceed systematically to fill in the circles based on the initial knowledge that $k = 2$ and $e = 5$:

$i = 10$ $(5, i)$

[Throughout the solution to this problem, we have used notation such as (5, 7, c) to indicate that there is a straight line joining the circles containing 5, 7 and c. Thus, $5 + 7 + c = 15$ or $c = 3$.]

$b = 8$ $(2, 5, b)$
$d = 7$ $(8, d)$
$c = 3$ $(5, 7, c)$
$a = 4$ $(a, 8, 3)$
$g = 9$ $(4, g, 2)$
$f = 6$ $(4, f, 5)$
$h = 1$ $(9, h, 5)$



We can verify that the numbers along each of the ten straight lines add to 15.

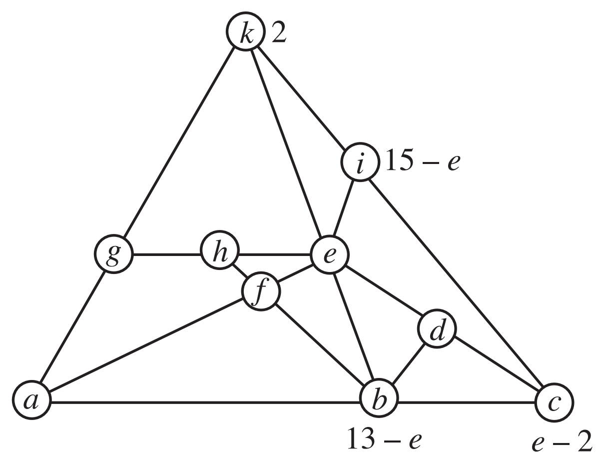
(b) (i) *Solution 1*

We start with knowing that $k = 2$ and e is unknown.

Then

$i = 15 - e$ (e, i)
$c = 15 - 2 - (15 - e) = e - 2$ $(2, 15 - e, c)$
$b = 15 - 2 - e = 13 - e$ $(2, e, b)$

Therefore, $b = 13 - e$ and $c = e - 2$.



Solution 2

We start with knowing that $k = 2$ and e is unknown.

Then

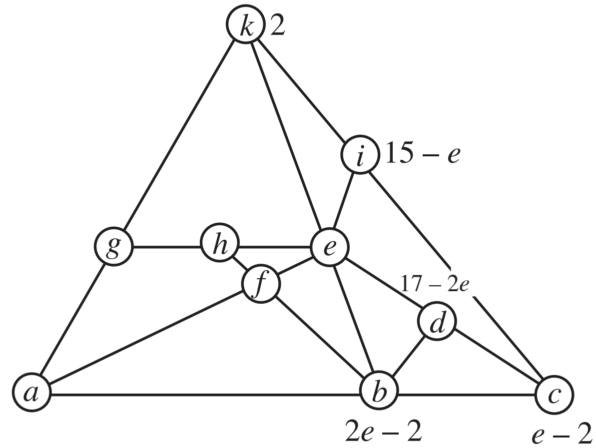
$i = 15 - e$ $\quad (e, i)$

$c = 15 - 2 - (15 - e) = e - 2$ $\quad (2, 15 - e, c)$

$d = 15 - e - (e - 2) = 17 - 2e$ $\quad (e, d, e - 2)$

$b = 15 - (17 - 2e) = 2e - 2$ $\quad (b, 17 - 2e)$

Therefore, $b = 2e - 2$ and $c = e - 2$.



(ii) *Solution*

We know from (i) that $k = 2$, $b = 13 - e$, and $c = e - 2$.

Therefore, $d = 15 - b = 15 - (13 - e) = 2 + e$.

But we also know that

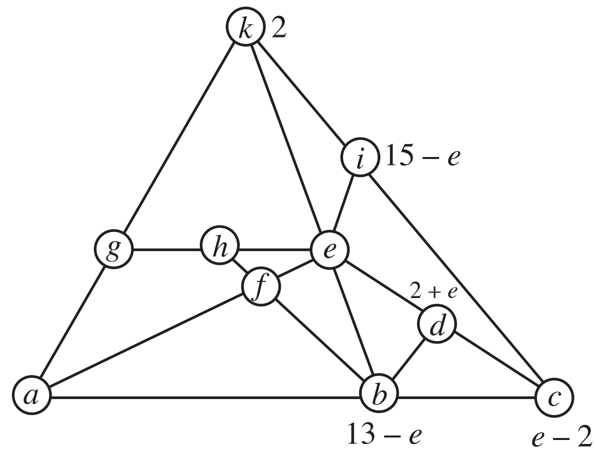
$$e + d + c = 15$$

$$e + (2 + e) + (e - 2) = 15$$

$$3e = 15$$

$$e = 5$$

Therefore, $e = 5$.



(c) *Solution*

We can model our approach from (b).

Starting with e being unknown and $k = x$, we express some of the circles in terms of x and e:

$i = 15 - e$ $\quad (e, i)$

$c = 15 - x - (15 - e) = e - x$ $\quad (x, 15 - e, c)$

$b = 15 - x - e$ $\quad (x, e, b)$

$d = 15 - (15 - x - e) = x + e$ $\quad (15 - x - e, d)$

But we know that

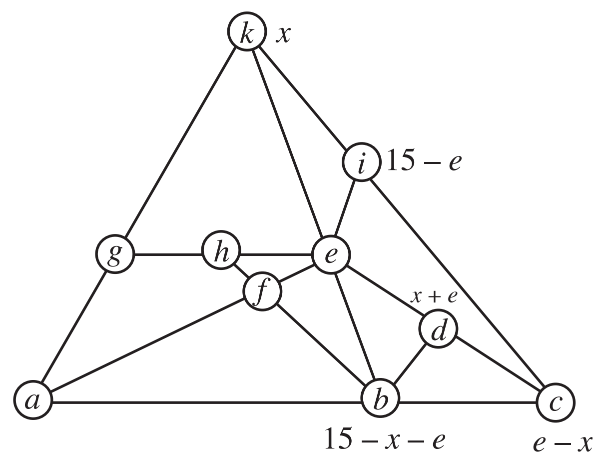
$$e + d + c = 15$$

$$e + (x + e) + (e - x) = 15$$

$$3e = 15$$

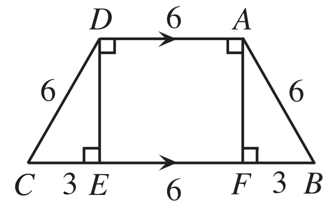
$$e = 5$$

Therefore, e must still be equal to 5.



2. (a) *Solution 1*

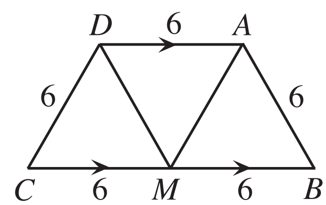
Drop perpendiculars from D and A to BC, meeting BC at E and F respectively.
Since DA is parallel to CB, then DE and AF are also perpendicular to DA.
Since $DAFE$ is a rectangle, then $EF = 6$.
Since $DC = AB$, $DE = AF$ and ΔDEC and ΔAFB are right-angled, then they are congruent triangles, and so $CE = BF$, and so both of these lengths must be equal to 3.



By the Pythagorean Theorem, $DE = \sqrt{DC^2 - CE^2} = \sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$.
Therefore, the sides of ΔDEC are in the ratio 1 to $\sqrt{3}$ to 2, so DEC is a 30-60-90 triangle, with $\angle DCE = 60^\circ$ and $\angle CDE = 30^\circ$.
Therefore, since we have congruent triangles, $\angle DCB = \angle ABC = \angle DCE = 60^\circ$ and $\angle CDA = \angle DAB = 90^\circ + \angle CDE = 120^\circ$.

Solution 2

Join D to the midpoint M of CB.
Then $CM = MB = 6$.
Since DM and MB are parallel and of equal length, then AB and DM will also be parallel and equal length.
Thus, $DM = 6$, and so ΔDCM is equilateral.

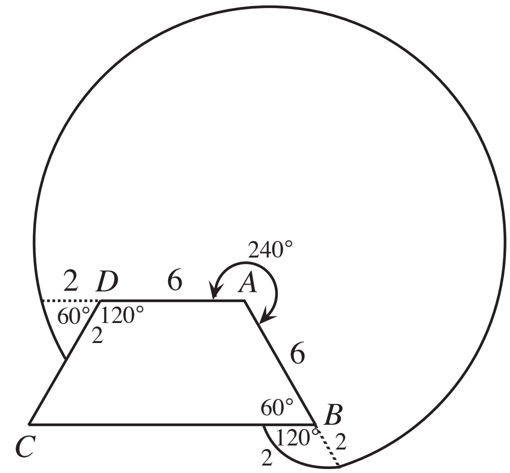


Therefore, $\angle DCB = \angle DCM = 60^\circ$. By symmetry, $\angle ABC = \angle DCB = 60^\circ$.
Since DA and CB are parallel, $\angle CDA = \angle DAB = 120^\circ$.

(b) (i) If Chuck was attached to a point P and there were no obstructions, he would be able to reach a circle of radius 8 m. (If Chuck stays at the end of his chain, he could trace out a circle of radius 8 m, but Chuck can move everywhere inside this circle, since his chain does not have to be tight.) However, here we have an obstruction – the trapezoidal barn.

Since the interior angle of the barn at point A is 120°, then the exterior angle of the barn is 240°. So Chuck can certainly reach the area which is a 240° sector of radius 8 m, centred at A. (If Chuck extends the chain as far as possible in a straight line in the direction of D from A, can then walk in a clockwise direction, keeping the chain at its full length until the chain lies along AB. He will have moved through 240°, and the region is the sector of a circle.)

However, when the chain is fully extended in the direction of D, Chuck will be 2 m past point D. He will thus be free to move towards side DC of the barn. If he does this and keeps the chain tight, he will trace out part of a circle of radius 2 m centred at D. (Point D now serves as a "pivot" point for the chain.) Since the exterior angle of the barn at point D is 240°, then the angle between AD extended and DC is 60°. Therefore, Chuck can reach a 60° sector of a circle of radius 2 m, centred at D.



When the chain is fully extended in the direction of B, Chuck will be 2 m past point B. He will thus be free to move towards side BC of the barn. If he does this and keeps the chain tight, he will trace out part of a circle of radius 2 m centred at B. (Point B now serves as a "pivot" point for the chain.) Since the exterior angle of the barn at point B is 300° (the interior angle at B is 60°), then the angle between AB extended and BC is 120°. Therefore, Chuck can reach a 120° sector of a circle of radius 2 m, centred at B.

The area of a sector of angle θ° of a circle of radius r is $\frac{\theta}{360}\pi r^2$.

Therefore, the total area that Chuck can reach is

$$\frac{240}{360}(\pi)8^2+\frac{60}{360}(\pi)2^2+\frac{120}{360}(\pi)2^2=\frac{2}{3}(64\pi)+\frac{1}{6}(4\pi)+\frac{1}{3}(4\pi)=\frac{128\pi}{3}+2\pi=\frac{134\pi}{3}$$

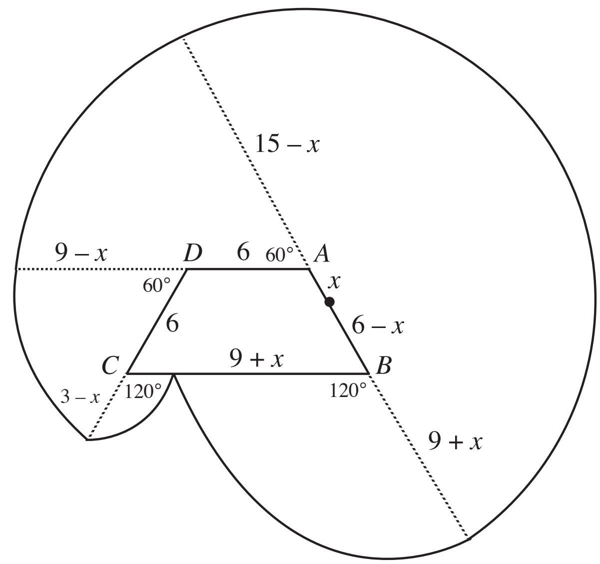
square metres.

(ii) Let x be the distance along AB from A to P.

Since the total perimeter of the barn is 30 m and Chuck is attached with a 15 m chain, then Chuck can reach the same point on the barn whether he wraps the chain around the barn in a clockwise direction or a counterclockwise direction. This point will move, however, as P moves. For example, if Chuck was attached at point A (ie. if $x=0$), then he could reach a point 3 m along CB from C towards B wrapping in either the clockwise or counterclockwise direction. If Chuck was attached at point B (ie. if $x=6$), he could reach the midpoint of CD in either direction. As point P moves from A towards B, this furthest point on the barn that Chuck can reach will slide along BC towards C and then up CD towards D. If P is at the midpoint of AB (ie. if $x=3$), the furthest point along the barn that he can reach will be point C. So in our analysis, we must be careful as to whether $0\le x\le 3$ or $3\le x\le 6$.

Regardless of the value of x, Chuck can certainly reach a 180° sector of a circle of radius 15 centred at P.

We start in the counterclockwise direction.
Also, regardless of the value of x, Chuck can reach a 60° sector of a circle of radius $15 - x$ centred at A (using A as the new pivot point for the chain).
Still regardless of the value of x, Chuck can reach a 60° sector of a circle of radius $9 - x = (15 - x) - 6$ centred at D (using D as the new pivot point for the chain).
If $3 \le x \le 6$, then $9 - x \le 6$, so Chuck cannot reach past point C.



If $0 \le x \le 3$, then $9 - x \ge 6$, so Chuck can reach past point C, and so can reach a 120° sector of a circle of radius $3 - x = (9 - x) - 6$ centred at C (using C as the new pivot point for the chain).
Next, we consider the clockwise direction.
Regardless of the value of x, Chuck can reach a 120° sector of a circle of radius $9 + x = 15 - (6 - x)$ (the distance from B to P is $6 - x$) centred at B (using B as the new pivot point for the chain).
If $0 \le x \le 3$, then $9 + x \le 12$, so Chuck cannot reach past point C.
If $3 \le x \le 6$, then $9 + x \ge 12$, so Chuck can reach past point C, and so can reach a 120° sector of a circle of radius $x - 3 = (9 + x) - 12$ centred at C (using C as the new pivot point for the chain).

We now calculate the total area that Chuck can reach.
If $0 \le x \le 3$, then the area that Chuck can reach is

$$\frac{180}{360}\pi 15^2 + \frac{60}{360}\pi(15 - x)^2 + \frac{60}{360}\pi(9 - x)^2 + \frac{120}{360}\pi(3 - x)^2 + \frac{120}{360}\pi(9 + x)^2$$

$$= \frac{1}{2}\pi(225) + \frac{1}{6}\pi\left(225 - 30x + x^2\right) + \frac{1}{6}\pi\left(81 - 18x + x^2\right) + \frac{1}{3}\pi\left(9 - 6x + x^2\right) + \frac{1}{3}\pi\left(81 + 18x + x^2\right)$$

$$= \frac{1}{6}\pi\left(675 + 225 - 30x + x^2 + 81 - 18x + x^2 + 18 - 12x + 2x^2 + 162 + 36x + 2x^2\right)$$

$$= \frac{1}{6}\pi\left(1171 - 24x + 6x^2\right)$$

$$= \pi x^2 - 4\pi x + \frac{387}{2}\pi$$

If $3 \le x \le 6$, then the area that Chuck can reach is

$$\frac{180}{360}\pi 15^2 + \frac{60}{360}\pi(15-x)^2 + \frac{60}{360}\pi(9-x)^2 + \frac{120}{360}\pi(9+x)^2 + \frac{120}{360}\pi(x-3)^2$$
$$= \pi x^2 - 4\pi x + \frac{387}{2}\pi$$

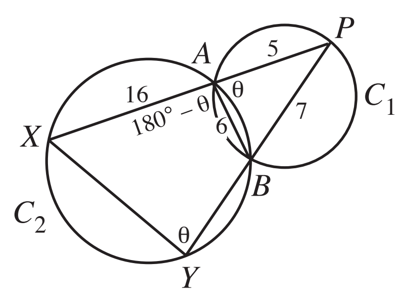
(Notice that the one term between these two initial expressions that seems to be different is actually the same!)

Therefore, no matter what the value of x is, the area that Chuck can reach is $\pi x^2 - 4\pi x + \frac{387}{2}\pi$. This is a parabola opening upwards, so the vertex of the parabola gives us the minimum of the parabola. This vertex is at $x = -\frac{-4\pi}{2(\pi)} = 2$. Since $x = 2$ is between the endpoints of the allowable interval (0 and 6), then this will give the minimum.

Therefore, the location of P which minimizes the area that Chuck can reach is 2 m along the wall from A towards B.

3. (a) *Solution 1*

Let $\angle PAB = \theta$.
Then $\angle XAB = 180^\circ - \theta$, and so $\angle XYB = \theta$ since $XYBA$ is a cyclic quadrilateral, and so opposite angles sum to 180°.
Therefore, ΔPAB is similar to ΔPYX (common angle at P, equal angle θ).
So, $\frac{XY}{BA} = \frac{PX}{PB}$ or $XY = \frac{BA \cdot PX}{PB} = \frac{6(5+16)}{7} = 18$.

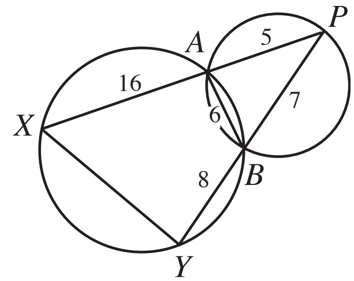


Solution 2

By the cosine law in ΔAPB,

$$AB^2 = PA^2 + PB^2 - 2(PA)(PB)\cos(\angle APB)$$
$$36 = 25 + 49 - 2(5)(7)\cos(\angle APB)$$
$$\cos(\angle APB) = \tfrac{38}{70} = \tfrac{19}{35}$$

Now, PX and PY are both secants of circle C_2, so by the Secant-Secant Theorem,



$$PA \cdot PX = PB \cdot PY$$
$$5(5+16) = 7(7+BY)$$
$$105 = 7(7+BY)$$
$$15 = 7 + BY$$
$$BY = 8$$

Now in ΔPXY we know the lengths of sides PX (length 21), PY (length 15) and the cosine of $\angle XPY = \angle APB$, so we can use the cosine law to calculate the length of XY.

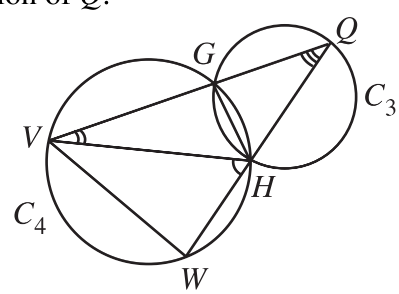
$$XY^2 = PX^2 + PY^2 - 2(PX)(PY)\cos(\angle XPY)$$
$$XY^2 = 441 + 225 - 2(21)(15)\left(\tfrac{19}{35}\right)$$
$$XY^2 = 441 + 225 - 2(3)(3)(19)$$
$$XY^2 = 441 + 225 - 342$$
$$XY^2 = 324$$
$$XY = 18$$

Therefore, the length of XY is 18.

(b) *Solution 1*

Since the circle C_4 is fixed, the length VW will be fixed if the angle it subtends on the circle is fixed, ie. if the angle $\angle VHW$ does not depend on the position of Q.

Now



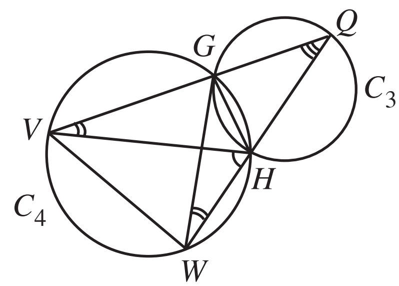
$$\begin{aligned}\angle VHW &= 180^\circ - \angle VHQ \\ &= 180^\circ - \left(180^\circ - \angle GVH - \angle GQH\right) \\ &= \angle GVH + \angle GQH\end{aligned}$$

But since chord GH is a chord of fixed length in both circles (ie. it doesn't change as Q moves), then the angles that it subtends in both circles are constant. In particular, $\angle GVH$ and $\angle GQH$ are both fixed (that is, they do not depend on the position of Q).

Since both of these angles are fixed, then $\angle VHW = \angle GVH + \angle GQH$ is also fixed.

Therefore, the length of VW is fixed.

Solution 2

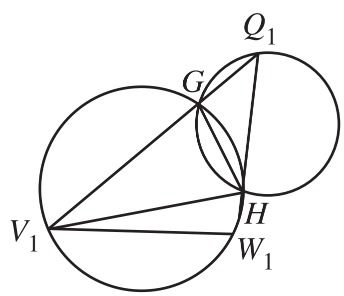


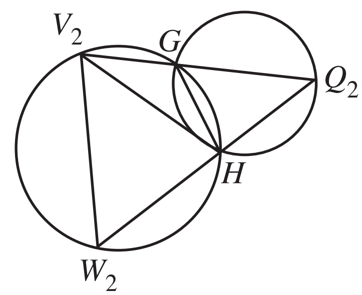
We start by noting that chord GH has a constant length, ie. does not depend on the the position of Q. Thus, GH is a constant chord in both C_3 and C_4.
In C_3, let $\angle GQH = \alpha$.
In C_4, let $\angle GVH = \angle GWH = \beta$.
These angles are constant since GH is of constant length.

Therefore, $\angle VHQ = \angle QGW = 180^\circ - (\alpha + \beta)$, and so $\angle VGW = \angle WHV = \alpha + \beta$.
Since these last two angles are constant for all position of Q, then VW is a chord of constant length.

Solution 3

We proceed by considering two different positions for the point Q, which we call Q_1 and Q_2. These points will create two different positions for the line segment VW, which we call V_1W_1 and V_2W_2.
To show that the length of VW is constant, we must show that V_1W_1 and V_2W_2 have the same length. We do know that the points G and H are fixed, so the length of GH does not vary.





As in part (a) Solution 1, ΔQ_1HG and $\Delta Q_1V_1W_1$ are similar triangles, as are ΔQ_2HG and $\Delta Q_2V_2W_2$. (The two positions for Q play the same role as P, points G and H play the same role as A and B, and the points V and W play the same role as X and Y.)
Therefore, by similar triangles, $\dfrac{V_1W_1}{HG} = \dfrac{Q_1V_1}{Q_1H}$ or $V_1W_1 = HG \cdot \dfrac{Q_1V_1}{Q_1H}$.
Also, by similar triangles, $\dfrac{V_2W_2}{HG} = \dfrac{Q_2V_2}{Q_2H}$ or $V_2W_2 = HG \cdot \dfrac{Q_2V_2}{Q_2H}$.
Since the length of HG (or GH) does not change, to show that V_1W_1 and V_2W_2 have the same length, we must show that $\dfrac{Q_1V_1}{Q_1H} = \dfrac{Q_2V_2}{Q_2H}$.

Join H to each of V_1 and V_2.
Now GH is a chord of fixed length in both circles, so the angle that it subtends at any point on the circumference of each circle is the same.

Therefore, $\angle GQ_1H = \angle GQ_2H$ and $\angle GV_1H = \angle GV_2H$.
But this tells us that ΔQ_1HV_1 and ΔQ_2HV_2 are similar.
Thus, $\dfrac{Q_1V_1}{Q_2V_2} = \dfrac{Q_1H}{Q_2H}$ or $\dfrac{Q_1V_1}{Q_1H} = \dfrac{Q_2V_2}{Q_2H}$.
Therefore, $V_1W_1 = HG \cdot \dfrac{Q_1V_1}{Q_1H} = HG \cdot \dfrac{Q_2V_2}{Q_2H} = V_2W_2$, ie. the length of VW is constant.

4. (a) *Solution 1*

Since a, b and c are the roots of the equation $x^3 - 6x^2 + 5x - 1 = 0$, then using the properties of the coefficients of a cubic equation,

$$\begin{aligned} a + b + c &= 6 \\ ab + ac + bc &= 5 \\ abc &= 1 \end{aligned}$$

Since we know that each of a, b and c is a roots of the equation, then

$$\begin{aligned} a^3 - 6a^2 + 5a - 1 &= 0 \\ b^3 - 6b^2 + 5b - 1 &= 0 \\ c^3 - 6c^2 + 5c - 1 &= 0 \end{aligned}$$

or after rearranging

$$\begin{aligned} a^3 &= 6a^2 - 5a + 1 \\ b^3 &= 6b^2 - 5b + 1 \qquad (*) \\ c^3 &= 6c^2 - 5c + 1 \end{aligned}$$

Adding these three equations, we obtain

$$\begin{aligned} a^3 + b^3 + c^3 &= 6a^2 + 6b^2 + 6c^2 - 5a - 5b - 5c + 3 \\ &= 6\left(a^2 + b^2 + c^2\right) - 5(a + b + c) + 3 \end{aligned}$$

We already know that $a + b + c = 6$, so if we could determine the value of $a^2 + b^2 + c^2$, then we would know the value of $a^3 + b^3 + c^3$.
But

$$\begin{aligned} (a + b + c)^2 &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \\ 6^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ac) \\ a^2 + b^2 + c^2 &= 36 - 2(5) \\ a^2 + b^2 + c^2 &= 26 \end{aligned}$$

and so

$$a^3 + b^3 + c^3 = 6\left(a^2 + b^2 + c^2\right) - 5(a + b + c) + 3 = 6(26) - 5(6) + 3 = 129$$

If we know take the equations in (*) and multiply both sides in the first, second and third equations by a, b and c, respectively, we get

$$a^4 = 6a^3 - 5a^2 + a$$
$$b^4 = 6b^3 - 5b^2 + b \qquad (**)$$
$$c^4 = 6c^3 - 5c^2 + c$$

which we then can add to obtain

$$\begin{aligned} a^4 + b^4 + c^4 &= 6\left(a^3 + b^3 + c^3\right) - 5\left(a^2 + b^2 + c^2\right) + (a + b + c) \\ &= 6(129) - 5(26) + 6 \\ &= 650 \end{aligned}$$

Repeating the process one more time by multiplying the first, second and third equations in (**) by a, b and c, respectively, and adding, we obtain

$$\begin{aligned} a^5 + b^5 + c^5 &= 6\left(a^4 + b^4 + c^4\right) - 5\left(a^3 + b^3 + c^3\right) + \left(a^2 + b^2 + c^2\right) \\ &= 6(650) - 5(129) + 26 \\ &= 3281 \end{aligned}$$

Therefore, the value of $a^5 + b^5 + c^5$ is 3281.

Solution 2

Since a, b and c are the roots of the equation $x^3 - 6x^2 + 5x - 1 = 0$, then using the properties of the coefficients of a cubic equation,

$$s = a + b + c = 6$$
$$t = ab + ac + bc = 5$$
$$p = abc = 1$$

We will attempt to express $a^5 + b^5 + c^5$ in terms of s, t and p, which will thus allow us to calculate the value of $a^5 + b^5 + c^5$.

First,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$
$$a^2 + b^2 + c^2 = s^2 - 2t$$

Next,

$$\left(a^2 + b^2 + c^2\right)(a + b + c) = a^3 + b^3 + c^3 + a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

$$\begin{aligned} a^3 + b^3 + c^3 &= s\left(s^2 - 2t\right) - \left[a^2b + a^2c + b^2a + b^2c + c^2a + c^2b\right] \\ &= s\left(s^2 - 2t\right) - \left[(ab + ac + bc)(a + b + c) - 3abc\right] \\ &= s\left(s^2 - 2t\right) - \left[ts - 3p\right] \\ &= s^3 - 3st + 3p \end{aligned}$$

We can now attempt to express $a^5 + b^5 + c^5$ as

$$
\begin{aligned}
&a^5+b^5+c^5\\
&=\left(a^2+b^2+c^2\right)\left(a^3+b^3+c^3\right)-\left[a^2b^3+a^2c^3+b^2a^3+b^2c^3+c^2a^3+c^2b^3\right]\\
&=\left(a^2+b^2+c^2\right)\left(a^3+b^3+c^3\right)-\left[\left(a^2b^2+a^2c^2+b^2c^2\right)(a+b+c)-\left(a^2b^2c+a^2bc^2+ab^2c^2\right)\right]\\
&=\left(s^2-2t\right)\left(s^3-3st+3p\right)-\left[\left[(ab+ac+bc)^2-2\left(a^2bc+ab^2c+abc^2\right)\right](a+b+c)-abc(ab+ac+bc)\right]\\
&=\left(s^2-2t\right)\left(s^3-3st+3p\right)-\left[\left[t^2-2abc(a+b+c)\right]s-pt\right]\\
&=\left(s^2-2t\right)\left(s^3-3st+3p\right)-\left[\left[t^2-2ps\right]s-pt\right]\\
&=\left(6^2-2(5)\right)\left(6^3-3(6)(5)+3(1)\right)-\left[\left[5^2-2(1)(6)\right](6)-1(5)\right]\\
&=(26)(129)-\left[[13](6)-5\right]\\
&=3354-[73]\\
&=3281
\end{aligned}
$$

Therefore, the value of $a^5+b^5+c^5$ is 3281.

(b) We will proceed by dividing our proof into several steps.

Step 1: Estimate the values of a, b and c

Step 2: Show that $a^n+b^n+c^n$ is an integer for every positive integer n

Step 3: Final conclusion

Step 1: Estimate the values of a, b and c

Define $f(x)=x^3-6x^2+5x-1$.

If $x<0$, then $x^3<0$, $-6x^2<0$, $5x<0$, and $-1<0$, so $f(x)=x^3-6x^2+5x-1<0$.

This tells us that $f(x)=0$ cannot have any negative roots. It is also clear that 0 is not a root of $f(x)=0$, so each of a, b and c is positive.

If we calculate a few values of $f(x)=x^3-6x^2+5x-1$, we obtain $f(0)=-1$, $f(1)=-1$, $f(2)=-7$, $f(3)=-13$, $f(4)=-13$, $f(5)=-1$, and $f(6)=29$.

Therefore, one of the roots is between 5 and 6.

However, we know from part (a) that $a+b+c=6$, so since all three roots are positive, then we must have $5<c<6$ and $0<a,b<1$. (Since one root is bigger than 5, each root is positive, and the sum of the three roots is 6, then neither a nor b is bigger than 1.)

Since $5<c<6$ and $a+b+c=6$, then $0<a+b<1$.

We also know from part (a) that $abc=1$. Since $5<c<6$, then $\frac{1}{6}<ab<\frac{1}{5}$ and since each of a and b is less than 1, then each of a and b must be bigger than $\frac{1}{6}$.

Since a and b are each bigger than $\frac{1}{6}$ and $0<a+b<1$, then $\frac{1}{6}<a,b<\frac{5}{6}$.

(We could have proceeded less formally by doing some quick calculations to see that $f(0.1)=-0.559$, $f(0.2)=-0.232$, $f(0.3)=-0.013$, $f(0.4)=0.104$, $f(0.5)=0.125$,

$f(0.6) = 0.056$, $f(0.7) = -0.097$, and so a must be between 0.3 and 0.4, and b must be between 0.6 and 0.7.)

Step 2: Show that $a^n + b^n + c^n$ is an integer for every positive integer n

In part (a), we saw that $a^n + b^n + c^n$ is definitely an integer for n equal to 1, 2 and 3.
If we return to the set of equations (*) in Solution 1 of part (a) and multiply the three equations by a^{n-3}, b^{n-3} and c^{n-3}, respectively, we obtain

$$\begin{aligned} a^n &= 6a^{n-1} - 5a^{n-2} + a^{n-3} \\ b^n &= 6b^{n-1} - 5b^{n-2} + b^{n-3} \qquad (***) \\ c^n &= 6c^{n-1} - 5c^{n-2} + c^{n-3} \end{aligned}$$

and adding, we get

$$a^n + b^n + c^n = 6\left(a^{n-1} + b^{n-1} + c^{n-1}\right) - 5\left(a^{n-2} + b^{n-2} + c^{n-2}\right) + \left(a^{n-3} + b^{n-3} + c^{n-3}\right) \quad (****)$$

for every n greater than or equal to 4.
If we set n equal to 4, then since $a^k + b^k + c^k$ is an integer for k equal to 1, 2 and 3, then by (****), $a^4 + b^4 + c^4$ is also an integer.
If we set n equal to 5, then since $a^k + b^k + c^k$ is an integer for k equal to 2, 3 and 4, then by (****), $a^5 + b^5 + c^5$ is also an integer.
It is now clear than we can continue this process inductively, since if $a^k + b^k + c^k$ is an integer for k equal to $n-3$, $n-2$, and $n-1$, then $a^n + b^n + c^n$ will also be an integer, by (****).
In particular, we can conclude that $a^{2003} + b^{2003} + c^{2003}$ and $a^{2004} + b^{2004} + c^{2004}$ are both integers, say $a^{2003} + b^{2003} + c^{2003} = M$ and $a^{2004} + b^{2004} + c^{2004} = N$.

Step 3: Final conclusion

Since each of a and b is between 0 and 1, then $a^{2003} > a^{2004}$ and $b^{2003} > b^{2004}$, so $a^{2003} + b^{2003} > a^{2004} + b^{2004}$.
Since a is less than $\frac{5}{6}$, then a is less than 0.9, so $a^2 < 0.81$, so $a^4 < (0.81)^2 < 0.7$, so $a^8 < (0.7)^2 < 0.5$, so $a^{16} < (0.5)^2 < 0.25$.
Similarly, since b is less than $\frac{5}{6}$, then $b^{16} < 0.25$.
Therefore, $a^{16} + b^{16} < 0.5$, and since each of a and b is less than 1, then $a^{2004} + b^{2004} < a^{2003} + b^{2003} < a^{16} + b^{16} < 0.5$.
Therefore, since $c^{2003} = M - \left(a^{2003} + b^{2003}\right)$ and $a^{2003} + b^{2003} < 0.5$, then the closest integer to c^{2003} is M and the distance between them is $a^{2003} + b^{2003}$. Similarly, the closest integer to c^{2004} is N and the distance between them is $a^{2004} + b^{2004}$.
But $a^{2004} + b^{2004} < a^{2003} + b^{2003}$, so c^{2004} is closer to N than c^{2003} is to M, as required.