The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

The
Canadian Open Mathematics Challenge

Wednesday, November 27, 2002

Solutions

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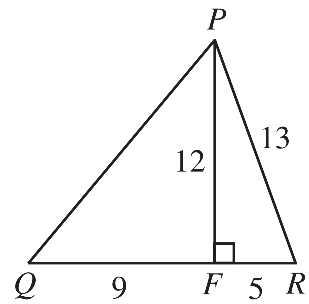
Part A

1. By Pythagoras in ΔPFR, $PF^2 = 13^2 - 5^2 = 144$, or $PF = 12$.
By Pythagoras in ΔPFQ, $PQ^2 = 9^2 + 12^2 = 225$, or $PQ = 15$.
Therefore, the side lengths of ΔPQR are 13, 14 and 15, i.e. the perimeter is 42.



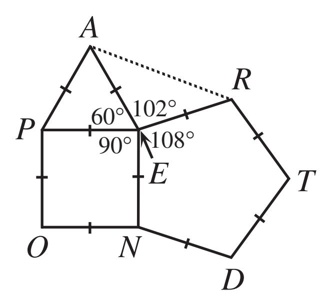
2. *Solution 1*

$$\begin{aligned} x^2 + 5xy + y^2 &= x^2 + 2xy + y^2 + 3xy \\ &= (x+y)^2 + 3xy \\ &= 4^2 + 3(-12) \\ &= -20 \end{aligned}$$

Solution 2
Examining the two given equations, we see that $x = 6$ and $y = -2$ is a solution.
Therefore, $x^2 + 5xy + y^2 = 6^2 + 5(6)(-2) + (-2)^2 = 36 - 60 + 4 = -20$.

Solution 3
We solve the first equation for x and substitute into the second equation.
From the first equation, $x = 4 - y$.
Substituting into the second equation, $(4-y)y = -12$ or $0 = y^2 - 4y - 12$.
Factoring, $0 = (y-6)(y+2)$, i.e. $y = 6$ or $y = -2$. The corresponding values of x are $x = -2$ and $x = 6$, which give the same answer as in Solution 2, i.e. $x^2 + 5xy + y^2 = -20$.

3. To determine $\angle EAR$, we look at the angles around the point E.
We know that $\angle AER + \angle REN + \angle NEP + \angle PEA = 360^\circ$.
Since $\angle PEA$ is an angle in an equilateral triangle, $\angle PEA = 60^\circ$.
Since $\angle NEP$ is an angle in a square, $\angle NEP = 90^\circ$.



Since $\angle REN$ is an angle in a regular pentagon, $\angle REN = \frac{1}{5}(540^\circ) = 108^\circ$.

Therefore,

$$\angle AER = 360^\circ - \angle REN - \angle NEP - \angle PEA$$
$$= 360^\circ - 108^\circ - 90^\circ - 60^\circ$$
$$= 102^\circ$$

Now since *PEA* is an equilateral triangle, *OPEN* is a square, and *TREND* is a regular pentagon, then their side lengths must all be the same, since *OPEN* and *TREND* share a side, and since *OPEN* and *PEA* share a side. In particular, $AE = ER$.

Therefore, ΔARE is an isosceles triangle, and so

$\angle ARE = \frac{1}{2}\left(180^\circ - \angle AER\right) = \frac{1}{2}\left(180^\circ - 102^\circ\right) = 39^\circ$.

4. *Solution 1*

The sum of the 3rd, 4th and 5th terms of the sequence is equal to the sum of the first five terms of the sequence minus the sum of the first two terms of the sequence.

Thus, the sum is $\left[5(5)^2 + 6(5)\right] - \left[5(2)^2 + 6(2)\right] = 155 - 32 = 123.$

Solution 2

We determine the first 5 terms in the sequence and then add up the 3rd, 4th and 5th terms.

From the formula given, the sum of the first 1 terms is 11.

This tells us that the first term is 11.

From the formula given, the sum of the first 2 terms is 32. Since the first term is 11, then the second term is 21.

Next, the sum of the first 3 terms is 63, and so the third term is 31, since the first two terms are 11 and 21. (We could use the fact that the *sum* of the first two terms is 32, instead.)

Next, the sum of the first 4 terms is 104, and so the fourth term is 41.

Lastly, the sum of the first 5 terms is 155, and so the fifth term is 51.

Therefore, the sum of the 3rd, 4th and 5th terms is $31 + 41 + 51 = 123$.

Solution 3

Since the sum of the first n terms has a quadratic formula, then the terms in the sequence have a common difference, i.e. The sequence is an arithmetic sequence.

Therefore, the sum of the 3rd, 4th and 5th terms is equal to three times the 4th term.

The 4th term is the sum of the first four terms minus the sum of the first three terms, i.e. $104 - 63 = 41$.

Thus, the sum of the 3rd, 4th and 5th terms is $3(41) = 123$.

5. *Solution 1*

Since the value of this expression is the same for every positive integer a, then we can find the value by substituting in $a=1$.

Thus,

$$\frac{[(2a-1)\nabla(2a+1)]}{[(a-1)\nabla(a+1)]}=\frac{[1\nabla 3]}{[0\nabla 2]}=\frac{1+2+3}{0+1+2}=\frac{6}{3}=2$$

Therefore, the value required is 2.

Solution 2

If a is a positive integer, the only integer between $2a-1$ and $2a+1$ is $2a$. Similarly, the only integer between $a-1$ and $a+1$ is a.

Thus,

$$\frac{[(2a-1)\nabla(2a+1)]}{[(a-1)\nabla(a+1)]}=\frac{(2a-1)+2a+(2a+1)}{(a-1)+a+(a+1)}=\frac{6a}{3a}=2$$

Therefore, the value required is 2.

6. Label the two ends of the mirrors U and W, as shown.

Since the initial beam is parallel to the mirror WV, then $\angle UAS = 30^\circ$. Since the angle of incidence equals the angle of reflection, then the reflected beam of light also makes an angle of 30° with the mirror UV.

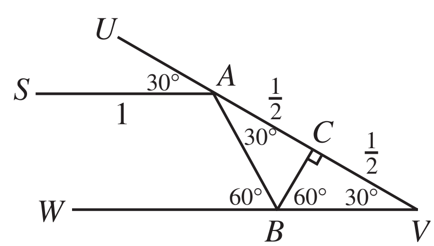
Let B be the point on the mirror WV that the light next strikes.

Since $\angle VAB = \angle AVB = 30^\circ$, then the angle of incidence, $\angle ABW$, is equal to 60°, because it is an external angle of ΔABV. ($\angle ABW$ could also have been calculated using the facts that $\angle SAB = 120^\circ$ and SA is parallel to WV.)

Therefore, the angle of reflection is also 60°.

Let C be the point on the mirror UV where the light next strikes. Since $\angle CVB = 30^\circ$ and $\angle VBC = 60^\circ$, then $\angle BCV = 90^\circ$. This tells us that the light is reflected straight back along its path from C back to S.

Therefore, the required distance is $2(SA+AB+BC)$.



Since we are given that $SA = AV = 1$, then since ΔABV is isosceles with BC an altitude, then $AC = CV = \frac{1}{2}$, and so $BC = \frac{1}{\sqrt{3}}(AC) = \frac{1}{2\sqrt{3}}$ and $AB = \frac{2}{\sqrt{3}}(AC) = \frac{1}{\sqrt{3}}$.

Therefore, the required distance is

$$2(SA+AB+BC) = 2\left(1+\frac{1}{\sqrt{3}}+\frac{1}{2\sqrt{3}}\right) = 2+\frac{3}{\sqrt{3}} = 2+\sqrt{3}$$

Thus, the total distance travelled by the beam is $\left(2+\sqrt{3}\right)$ m, or about 3.73 m.

7. *Solution 1*

Since P is formed by adding a 1 at the end of N, then $P = 10N + 1$.
Since Q is formed by adding a 1 in front of the 5 digits of N, then $Q = 100000 + N$.
Since $P = 3Q$,

$$10N + 1 = 3(100000 + N)$$
$$10N + 1 = 300000 + 3N$$
$$7N = 299999$$
$$N = 42857$$

Therefore, N is 42857.

Solution 2

Suppose N has digits *abcde*. Then since $P = 3Q$, we have $abcde1 = 3(1abcde)$.
Since the units digit on the left side is 1, then the units digit on the right is also 1, which means that $e = 7$.
Thus, $abcd71 = 3(1abcd7)$. Since the tens digit on the left side is 7 and we get a "carry" of 2 from multiplying the last digit on the right side by 3, then $3 \times d$ has a units digit of 5, i.e. $d = 5$.
Thus, $abc571 = 3(1abc57)$. Since the hundreds digit on the left side is 5 and we get a carry of 1 from multiplying the last two digits on the right side by 3, then the units digit of $3 \times c$ must be a 4, i.e. $c = 8$.
Thus, $ab8571 = 3(1ab857)$. In a similar fashion, we see that $b = 2$ and $a = 4$.
Therefore, $N = 42857$.

8. We are not told that M must be a *positive* integer, but it makes sense to look for a positive integer M that satisfies these conditions, since we want the *maximum* possible value of M.
Since there are 1000 numbers in the set $\{1,2,3,\ldots,999,1000\}$ and the probability that an x chosen randomly from this set is a divisor of M is $\frac{1}{100}$, then M must have 10 divisors between 1 and 1000.
Since we are told that $M \leq 1000$, then M must have exactly 10 positive divisors.
Therefore, M must be of the form p^9 where p is a prime number, or p^4q where p and q are both primes.
(Recall that to find the number of positive divisors of M, we find the prime factorization of M and then take each of the exponents, add 1, and find the product of these numbers. For example, if $M = 48 = 2^4 3$, then the number of positive divisors is $(4+1)(1+1) = 10$.)
Now, we want to determine the maximum M in each of these two forms.

<u>Case 1</u> $M = p^9$

Since $M \leq 1000$, then we must have $p = 2$, i.e. $M = 512$.
(If $p = 3$, then $p^9 = 19683$ is too large.)

Case 2 $M = p^4 q$

Since $M \le 1000$ and $5^4 = 625$, then we must have $p = 2$ or $p = 3$.

If $p = 2$, then the largest q can be so that q is prime and $M \le 1000$ is 61, i.e. $M = (16)(61) = 976$.

If $p = 3$, then the largest q can be so that q is prime and $M \le 1000$ is 11, i.e. $M = (81)(11) = 891$.

Therefore, the maximum possible value of M is 976.

Part B

1. (a) The slope of the line through P and F is

$$\frac{5-1}{0-8} = -\frac{1}{2}$$

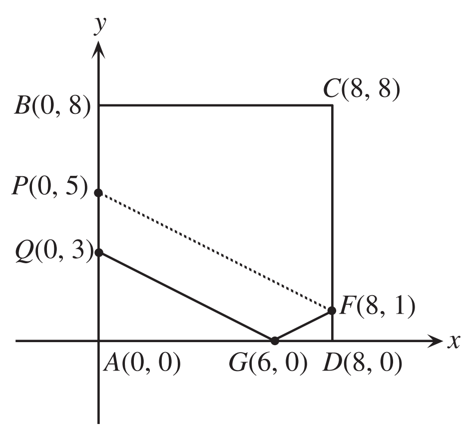
and so the slope of the desired line is also $-\frac{1}{2}$.

Since the point Q lies on the y-axis, then the y-intercept of the line is 3.

Therefore, the line is $y = -\frac{1}{2}x + 3$.

(b) Since AD lies along the x-axis, then G is the point where the line from (a) crosses the x-axis. To find the coordinates of G, we set $y = 0$ in the line from (a) to get $0 = -\frac{1}{2}x + 3$ or $x = 6$.

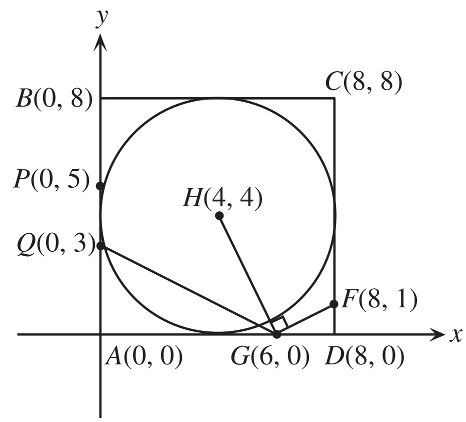
Therefore, the desired line passes through the points $G(6,0)$ and $F(8,1)$. Thus its slope is $\frac{1-0}{8-6} = \frac{1}{2}$, and so its equation is $y - 0 = \frac{1}{2}(x-6)$ or $y = \frac{1}{2}x - 3$.



(c) Since FG has slope $\frac{1}{2}$, then a line perpendicular to FG has slope -2, the negative reciprocal of $\frac{1}{2}$.

Since the desired line passes through $H(4,4)$, it has equation $y - 4 = -2(x-4)$ or $y = -2x + 12$.

(d) The circle has centre $H(4,4)$, and it is tangent to all four sides of the square, and so its radius must be 4, since the distance from the centre to each of the four sides is 4.
Does this circle intersect the line $y = \frac{1}{2}x - 3$, i.e. the line through F and G?



We must find the shortest distance between the centre of the circle and the line, i.e. the perpendicular distance. We already have the equation of a line through H that is perpendicular to the line through F and G, the line $y = -2x + 12$. Where do these lines intersect? Setting y-coordinates equal,

$$\frac{1}{2}x - 3 = -2x + 12$$
$$\frac{5}{2}x = 15$$
$$x = 6$$

i.e. the lines intersect at the point $G(6,0)$! Therefore, the shortest distance from H to the line through F and G is the distance from H to G, which is

$$\sqrt{(6-4)^2 + (4-0)^2} = \sqrt{20}$$

i.e. is greater than than $4 = \sqrt{16}$, the radius of the circle.
Therefore, the circle does not intersect the line.

2. (a) For the product $(2A5)(13B)$ to be divisible by 36, we need it to be divisible by both 4 and 9. Since $2A5$ is odd, it does not contain a factor of 2.
Therefore, $13B$ must be divisible by 4.
For a positive integer to be divisible by 4, the number formed by its last two digits must be divisible by 4, i.e. $3B$ is divisible by 4, i.e. $B = 2$ or $B = 6$.

<u>Case 1</u> $B = 2$
In this case, 132 is divisible by 3, but not by 9. Therefore, for the original product to be divisible by 9, we need $2A5$ to be divisible by 3.
For a positive integer to be divisible by 3, the sum of its digits is divisible by 3, i.e. $2 + A + 5 = A + 7$ is divisible by 3.
Therefore, $A = 2$ or 5 or 8.

<u>Case 2</u> $B = 6$
In this case, 136 contains no factors of 3, so for the original product to be divisible by 9, we need $2A5$ to be divisible by 9.

For a positive integer to be divisible by 9, the sum of its digits is divisible by 9, i.e. $2 + A + 5 = A + 7$ is divisible by 9. Therefore, $A = 2$.

Therefore, the four possible ordered pairs are $(A,B) = (2,2),\ (8,2),\ (5,2),\ (2,6)$.

(b) (i) If $10a + b = 7m$, then $b = 7m - 10a$. Thus,

$$a - 2b = a - 2(7m - 10a) = 21a - 14m = 7(3a - 2m)$$

Since $3a - 2m$ is an integer, then by definition, $a - 2b$ is divisible by 7.

(ii) *Solution 1*

If $5c + 4d$ is divisible by 7, then $5c + 4d = 7k$ for some integer k.

Therefore, $d = \frac{1}{4}(7k - 5c)$.

So $4c - d = 4c - \frac{1}{4}(7k - 5c) = \frac{1}{4}(21c - 7k) = \dfrac{7(3c - k)}{4}$.

Since $4c - d$ is an integer, then $7(3c - k)$ must be divisible by 4. But 4 has no common factors with 7, so 4 must divide into $3c - k$, i.e. $\dfrac{3c - k}{4}$ is an integer.

Therefore, $4c - d = 7\left(\dfrac{3c - k}{4}\right)$, i.e. $4c - d$ is divisible by 7.

Solution 2

We note that $4c - d = (14c + 7d) - 2(5c + 4d)$.

Since both terms on the right side are divisible by 7, then $4c - d$ is divisible by 7.

Solution 3

Multiplying the expression $4c - d$ by 5 does not affect its divisibility by 7.

Thus, we can consider whether or not $20c - 5d$ is divisible by 7, and this will be equivalent to considering $4c - d$.

Since we are told that $5c + 4d = 7t$ for some integer t, then we know that $4(5c + 4d) = 20c + 16d = 28t$ or $20c = 28t - 16d$.

If we now consider $20c - 5d$, we see

$$\begin{aligned} 20c - 5d &= (28t - 16d) - 5d \\ &= 28t - 21d \\ &= 7(4t - 3d) \end{aligned}$$

Since $20c - 5d$ is divisible by 7 by definition, then $4c - d$ is divisible by 7.

3. (a) We consider the possible cases. On his first turn, Alphonse can take either 1 marble or 2 marbles.
If Alphonse takes 1 marble, Beryl can take 2 marbles and then Colleen 1 marble, to leave Alphonse with 1 marble left in the bowl. Therefore, Alphonse loses. (Note that Beryl and Colleen can agree on their strategy before the game starts.)
If Alphonse takes 2 marbles, Beryl can take 1 marble and then Colleen 1 marble, to leave Alphonse again with 1 marble left in the bowl. Therefore, Alphonse loses.
In either case, Beryl and Colleen can work together and force Alphonse to lose.

(b) *Solution 1*
On their two consecutive turns, Beryl and Colleen remove in total 2, 3 or 4 marbles. On his turn, Alphonse removes either 1 marble or 2 marbles. Therefore, by working together, Beryl and Colleen can ensure that the total number of marbles removed on any three consecutive turns beginning with Alphonse's turn is 4 or 5. (Totals of 3 and 6 cannot be guaranteed because of Alphonse's choice.)

Therefore, if N is a number of marbles in which Alphonse can be forced to lose, then so are $N+4$ and $N+5$, because Beryl and Colleen can force Alphonse to choose from N marbles on his second turn.

From (a), we know that 5 is a losing position for Alphonse. Also, 1 is a losing position for Alphonse. (Since 1 is a losing position, then 5 and 6 are both losing positions, based on our earlier comment.)

Since 5 and 6 are losing positions, then we can determine that 9, 10 and 11 are also losing positions, as are 13, 14, 15, and 16. If we add 4 to each of these repeatedly, we see that N is a losing position for every $N \geq 13$.

What about the remaining possibilities, i.e. 2, 3, 4, 7, 8, and 12?
For $N = 2$ or $N = 3$, if Alphonse chooses 1 marble, then either Beryl or Colleen is forced to take the last marble, so these are not losing positions for Alphonse, i.e. they are winning positions.
For $N = 4$, if Alphonse chooses 2 marbles, then either Beryl or Colleen is forced to take the last marble, so this is also not a losing position for Alphonse.

Next, we notice that if Alphonse chooses 1 marble, then the total number of marbles chosen by the three players will be 3, 4 or 5, and if Alphonse chooses 2 marbles, then the total number chosen will be 4, 5 or 6.
So if $N = 7$, then Alphonse can choose 1 marble and ensure that he receives 2, 3 or 4 marbles on his next turn. So 7 is a winning position for Alphonse.

If $N = 8$, then Alphonse can choose 2 marbles and ensure that he receives 2, 3 or 4 marbles on his next turn. So 8 is also a winning position for Alphonse.

Lastly, we consider $N = 12$.
If Alphonse chooses 1 marble, Beryl and Colleen can choose 1 each and return 9 marbles to Alphonse. As we have shown, this is a losing position for Alphonse.
If Alphonse chooses 2 marbles, Beryl and Colleen can choose 2 each and return 6 marbles to Alphonse. This is a losing position for Alphonse.

Therefore, the values of N for which Beryl and Colleen can force Alphonse to lose are 1, 5, 6, and all N for which $N \geq 9$.

Solution 2
First, we notice that if Alphonse chooses 1 marble, then the total number of marbles chosen by the three players will be 3, 4 or 5, and if Alphonse chooses 2 marbles, then the total number chosen will be 4, 5 or 6.

We define a "losing position" to be a number of marbles in the bowl so that if Alphonse starts with this number, he can be forced to lose.

From (a), we know that 5 is a losing position for Alphonse. Also, 1 is a losing position for Alphonse.
For $N = 2$ or $N = 3$, if Alphonse chooses 1 marble, then either Beryl or Colleen is forced to take the last marble, so these are not losing positions (ie. they are winning positions) for Alphonse.
For $N = 4$, if Alphonse chooses 2 marbles, then either Beryl or Colleen is forced to take the last marble, so this is a winning position for Alphonse.

How can we ensure that a starting position $N \geq 6$ is *not* a losing position?
N will not be a losing position if either none of $N-3$, $N-4$ or $N-5$ are losing positions, or none of $N-4$, $N-5$ or $N-6$ are losing positions. (If either group consists of three non-losing positions, then Alphonse can ensure that he gets a position from the appropriate set at the beginning of his next turn by choosing either 1 or 2 marbles respectively.)

Also, N will be a losing position as long as at least one of $N-3$, $N-4$, $N-5$ and at least one of $N-4$, $N-5$, $N-6$ are losing positions. (If there is a losing position in each group of 3, then no matter whether Alphonse chooses 1 or 2 marbles, then Beryl and Colleen will be able to force Alphonse into one of these previously known losing positions.)

Using these two criteria for checking whether a position is a losing position or not a losing position, we can see

i) $N = 6$ is a losing position, since $N - 5 = 1$ is a losing position in both groups of three
ii) $N = 7$ is not a losing position, since $N - 3$, $N - 4$, $N - 5$ (namely, 4, 3, 2) are not losing positions.
iii) $N = 8$ is not a losing position, since $N - 4$, $N - 5$, $N - 6$ (namely, 4, 3, 2) are not losing positions.
iv) $N = 9$ is a losing position, since $N - 4 = 5$ is a losing position in both groups of three
v) $N = 10$ is a losing position, since $N - 5 = 5$ is a losing position in both groups of three
vi) $N = 11$ is a losing position, since $N - 5 = 6$ is a losing position in both groups of three
vii) $N = 12$ is a losing position, since $N - 3 = 9$ and $N - 6 = 6$ are both losing positions

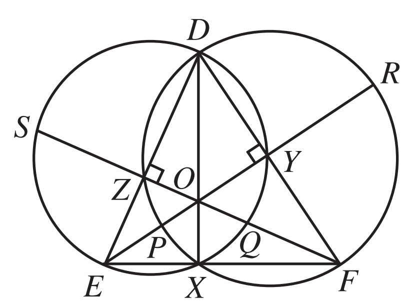
And so we have obtained 4 consecutive losing positions, which guarantees us that any $N \geq 13$ will also be a losing position, since $N - 4$ will be a losing position in both groups of 3.

Therefore, the values of *N* for which Beryl and Colleen can force Alphonse to lose are 1, 5, 6, and all *N* for which $N \geq 9$.

But among the first eight possibilities, there are now no more sets of three consecutive non-losing positions. This tells us that every position for $N \geq 9$ is a losing position, since we cannot find three consecutive non-losing positions as described above.

Therefore, the values of *N* for which Beryl and Colleen can force Alphonse to lose are 1, 5, 6, and all *N* for which $N \geq 9$.

4. *Solution 1*



Join *E* to *P*, *Y* and *R*, and join *F* to *Q*, *Z* and *S*.
Let *O* be the point of intersection of *EY* and *FZ*.
Since *EY* and *FZ* are altitudes in ΔDEF, then the third altitude, *DX* say, passes through *O*.
If we look at altitude *DX*, we see that $\angle DXE = 90^\circ$.
Since circle C_2 has *DE* as its diameter, then point *X* must lie on circle C_2, since a right angle is subtended by the diameter at point *X*.

Similarly, point *X* lies on circle C_1.
Therefore, *DX* is a chord of both circle C_1 and circle C_2.
We can now use the "Chord-Chord Theorem" in each of circle C_1 and C_2, to say

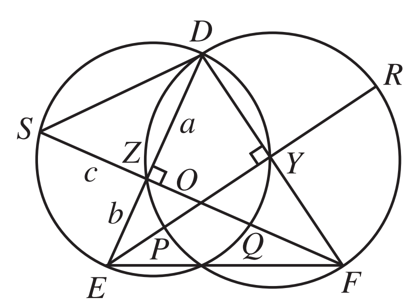
$SO \cdot OQ = DO \cdot OX$ (from circle C_2)
$RO \cdot OP = DO \cdot OX$ (from circle C_1)
From this we can conclude that $SO \cdot OQ = RO \cdot OP$.

Why does this allow us to conclude that P, Q, R, and S lie on the same circle?
From the equation, we obtain $\frac{SO}{OP} = \frac{RO}{OQ}$, which tells us that ΔSOP is similar to ΔROQ, and so $\angle PSQ = \angle PSO = \angle ORQ = \angle PRQ$.
Since the chord PQ subtends the equal angles $\angle PSQ$ and $\angle PRQ$ (in an undrawn circle), then the points P, Q, R, and S are concyclic.

Solution 2
In order to show that the four points lie on a circle, we will show that the points are equidistant from a fifth point, which will thus be the centre of the circle on which the four points lie.
Consider first the points Q and S. Any point equidistant from Q and S lies on the perpendicular bisector of the line joining these points. Since Q and S both lie on circle C_2, DE is a diameter of C_2, and QS is perpendicular to DE (since they lie on an altitude of the triangle), then DE is the perpendicular bisector of QS.
Similarly, DF is the perpendicular bisector of PR.
Therefore, any point that is equidistant from all four of the given points must lie on both DE and DF. Thus, the only possible candidate is point D. (And we already know that $DS = DQ$ and $DP = DR$ from our discussion of perpendicular bisectors.)

Thus, if we can show that $DS = DR$, then we will have shown what we need to show.

Method 1
Let $SZ = c$, $DZ = a$ and $EZ = b$.
Then $DS^2 = DZ^2 + SZ^2 = c^2 + a^2$ (Pythagoras).
Now if we extract ΔDSE, we see that $\angle DSE = 90^\circ$, since DE is a diameter of circle C_2. Therefore, ΔDSZ is similar to ΔSEZ, or $\frac{DZ}{SZ} = \frac{SZ}{EZ}$ or $c^2 = ab$.
Thus, $DS^2 = a^2 + ab = a(a+b) = DZ \cdot DE$.
Similarly, $DR^2 = DY \cdot DF$, looking at ΔDRF.



Now consider the points E, Z, Y, and F. Since $\angle EZF = \angle EYF = 90^\circ$, then EF must be the diameter of the circle containing points Y and Z (and points E and F).
Therefore, DE and DF are secants of the circle which intersect the circle at Z and Y, respectively. By the "Secant-Secant Theorem", $DZ \cdot DE = DY \cdot DF$.

From above, we can conclude that $DS^2 = DR^2$, or $DS = DR$, and thus $DP = DQ = DR = DS$.

Method 2

As above, we can obtain that $DS^2 = a^2 + ab = a(a+b) = DZ \cdot DE$.

Since $\angle DZF = 90^\circ$, then $DZ = DF\cos(\angle ZDF) = DF\cos(\angle EDF)$, and so $DS^2 = DZ \cdot DE = DE \cdot DF\cos(\angle EDF)$.

Repeating the process on the other side of the triangle gives us that $DR^2 = DY \cdot DF = DF \cdot DE\cos(\angle EDF)$, or $DR^2 = DS^2$.

Therefore, $DP = DQ = DR = DS$.

Therefore, we can conclude that the points P, Q, R, and S are concyclic.