Canadian Open Mathematics Challenge

Comments on the Paper

Part A

1. An operation "Δ" is defined by $a \Delta b = 1 - \frac{a}{b}$, $b \neq 0$.
   What is the value of $(1 \Delta 2) \Delta (3 \Delta 4)$?

   Solution
   By the definition of "Δ"
   $$1 \Delta 2 = 1 - \frac{1}{2} = \frac{1}{2}$$
   $$3 \Delta 4 = 1 - \frac{3}{4} = \frac{1}{4}$$
   and so $(1 \Delta 2) \Delta (3 \Delta 4) = \left(\frac{1}{2}\right) \Delta \left(\frac{1}{4}\right) = 1 - \frac{\frac{1}{2}}{\frac{1}{4}} = 1 - 2 = -1$

   ANSWER: –1

2. The sequence 9, 18, 27, 36, 45, 54, … consists of successive multiples of 9. This sequence is then altered by multiplying every other term by –1, starting with the first term, to produce the new sequence $-9, 18, -27, 36, -45, 54, \ldots$. If the sum of the first n terms of this new sequence is 180, determine n.

   Solution
   The terms in the sequence are paired, by combining each odd-numbered term with the next term (that is, we combine terms 1 and 2, 3 and 4, 5 and 6, etc).

   The sum of each of these pairs is 9.

   So we need 20 of these pairs to reach a sum of 180.

   Thus we need 2×20 or 40 terms.

   ANSWER: 40

3. The symbol $n!$ is used to represent the product $n(n-1)(n-2)\ \cdots\ (3)(2)(1)$.
For example, $4! = 4(3)(2)(1)$. Determine n such that $n! = (2^{15})(3^6)(5^3)(7^2)(11)(13)$.

Solution
Since $n!$ has a prime factor of 13, n must be at least 13.
Since $n!$ has no prime factor of 17, n must be less than 17.
These two facts are true because if $m \le n$, then m divides $n!$.
Since $n!$ has 5^3 as a factor, then $n \ge 15$, since we need $n!$ to have 3 factors which are multiples of 5.
We must thus determine if $n = 15$ or $n = 16$.
So we look at the number of factors of 2 in $16!$.
$16!$ gets 1 factor of 2 each from 2, 6, 10, 14
2 factors of 2 each from 4, 12
3 factors of 2 from 8
4 factors of 2 from 16
We have a total of 15 twos which then corresponds to $n = 16$.

ANSWER: 16

4. The symbol $\lfloor x \rfloor$ means the greatest integer less than or equal to x. For example,

$$\lfloor 5.7 \rfloor = 5,\ \lfloor \pi \rfloor = 3 \text{ and } \lfloor 4 \rfloor = 4.$$

Calculate the value of the sum

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \lfloor \sqrt{4} \rfloor + \ \cdots\ + \lfloor \sqrt{48} \rfloor + \lfloor \sqrt{49} \rfloor + \lfloor \sqrt{50} \rfloor.$$

Solution
We note that for k a positive integer and $k^2 \le n < (k+1)^2$, then $k \le \sqrt{n} < k+1$ and so $\lfloor \sqrt{n} \rfloor = k$.
Thus for $1 \le n \le 3, \lfloor \sqrt{n} \rfloor = 1$
$4 \le n \le 8, \lfloor \sqrt{n} \rfloor = 2$
$9 \le n \le 15, \lfloor \sqrt{n} \rfloor = 3$
etc.
So the sum equals

$$\begin{aligned} &(1+1+1)+(2+2+2+2+2)+\ (3+\cdots+3)+\cdots+(6+\cdots+6)+(7+7) \\ &= 3(1)+5(2)+7(3)+9(4)+11(5)+13(6)+2(7) \\ &= 3+10+21+36+55+78+14 \\ &= 217 \end{aligned}$$

ANSWER: 217

5. How many five-digit positive integers have the property that the product of their digits is 2000?

Solution
Let a five-digit number have the form $\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}\ \underline{e}$ where $0 \le a, b, c, d, e \le 9$, $a \ne 0$.
Since the product of the digits is 2000, we must have the product $abcde = 2000 = 2^4 5^3$.

Since the product of the digits is 2000, then 3 of the digits have to be 5. The remaining 2 digits must have a product of 16 or 2^4.

Thus the two remaining digits must be 4 and 4, or 2 and 8.

Possibility 1

Case 1 Using the numbers 5, 5, 5, 4, 4 there are $\frac{5!}{3!\,2!} = 10$ possible numbers.

Case 2 Using the numbers 5, 5, 5, 2, 8 there are $\frac{5!}{3!} = 20$ possible numbers.

There are 30 possible such numbers.

OR

Possibility 2

We choose 3 of the 5 positions for the "5s" in $\binom{5}{3}$ ways; there are 3 possibilities for the remaining two digits (including order): 2, 8; 4, 4; 8, 2.

So there are $3\times\binom{5}{3} = 3\times 10 = 30$ possible such 5 digit numbers.

ANSWER: 30

6. Solve the equation $4\left(16^{\sin^2 x}\right) = 2^{6\sin x}$, for $0 \le x \le 2\pi$.

Solution

We write all factors as powers of 2. Thus

$$4\left(16^{\sin^2 x}\right) = 2^{6\sin x}$$

$$2^2\left(2^{4\sin^2 x}\right) = 2^{6\sin x}$$

$$2^{4\sin^2 x+2} = 2^{6\sin x} \qquad (*)$$

Equating exponents (which we can do by taking base 2 logarithms),

$$4\sin^2 x + 2 = 6\sin x$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin x - 1)(\sin x - 1) = 0$$

Therefore, $\sin x = \frac{1}{2}$ or $\sin x = 1$.

Since $0 \le x \le 2\pi$, $x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{\pi}{2}$.

ANSWER: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

7. The sequence of numbers …, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, … is defined by $a_n - (n+1)a_{2-n} = (n+3)^2$, for all integers n. Calculate a_0.

Solution

Using the given general equation, we note that there are only two choices of n which yield equations containing a_2, $n = 0$ or 2.

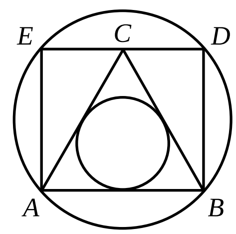
i.e. $a_0 - a_2 = 9$ from $n = 0$

$a_2 - 3a_0 = 25$ from $n = 2$

Adding the first equation to the second, we obtain $-2a_0 = 34$, so $a_0 = -17$.

ANSWER: –17

8. In the diagram, ΔABC is equilateral and the radius of its inscribed circle is 1. A larger circle is drawn through the vertices of the rectangle $ABDE$. What is the diameter of the larger circle?

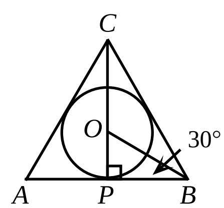


Solution

First, we calculate the side length of the equilateral triangle ABC.

Let O be the centre of the smaller circle and P the point of tangency of the circle to the side AB.

Join OP and OB. Then $\angle OPB = 90°$ by tangency and $\angle OBP = 30°$ by symmetry since $\angle CBA = 60°$.



Since $OP = 1$ and ΔBOP is 30°-60°-90°, then $OB = 2$ and $BP = \sqrt{3}$. Thus $AB = 2\sqrt{3}$.

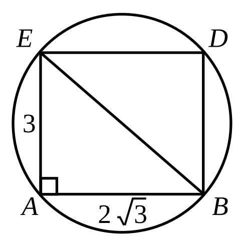
Also by symmetry, $CO = OB = 2$, so $CP = 3$.

Since $ABDE$ is a rectangle and $CP \perp AB$, then $AE = 3$.

We now look at the rectangle $ABDE$ and its circumcircle.

Since $ABDE$ is a rectangle, $\angle EAB = 90°$.

So BE is a diameter.



By Pythagoras,

$$BE^2 = EA^2 + AB^2$$
$$= 3^2 + \left(2\sqrt{3}\right)^2$$
$$= 21$$

The diameter is $\sqrt{21}$.

ANSWER: $\sqrt{21}$

Part B

1. Triangle ABC has vertices $A(0, 0)$, $B(9, 0)$ and $C(0, 6)$. The points P and Q lie on side AB such that $AP = PQ = QB$. Similarly, the points R and S lie on side AC so that $AR = RS = SC$. The vertex C is joined to each of the points P and Q. In the same way, B is joined to R and S.
   (a) Determine the equation of the line through the points R and B.
   (b) Determine the equation of the line through the points P and C.
   (c) The line segments PC and RB intersect at X, and the line segments QC and SB intersect at Y. Prove that the points A, X and Y lie on the same straight line.

Solution

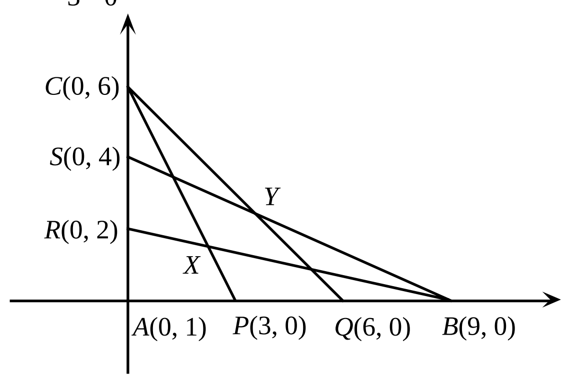
Since $A(0, 0)$, $B(9, 0)$ and $AP = PQ = QB$, then P has coordinates $(3, 0)$ and Q has coordinates $(6, 0)$.
Similarly, R is the point (0, 2) and S is the point (0, 4).

(a) Since R is (0, 2) and B is (9, 0), then the slope of RB is $m = \frac{2-0}{0-9} = -\frac{2}{9}$ and so the equation of the line is

$$y - 2 = -\frac{2}{9}(x - 0)$$
$$y = -\frac{2}{9}x + 2$$

(b) Since P is (3, 0) and C is (0, 6), then the slope of PC is $m = \frac{0-6}{3-0} = -2$ and so the equation of the line is

$$y - 0 = -2(x - 3)$$
$$y = -2x + 6$$



(c) First, we determine the coordinates of X.
Equating the lines from (a) and (b), we have

$$-\frac{2}{9}x + 2 = -2x + 6$$
$$\frac{16}{9}x = 4$$
$$x = \frac{9}{4}$$

and substituting into $y = -2x + 6 = -2\left(\frac{9}{4}\right) + 6 = \frac{3}{2}$, so X is the point $\left(\frac{9}{4}, \frac{3}{2}\right)$.

We calculate the equations of the lines QC and SB as in (a) and (b).
For QC, slope = $\frac{0-6}{6-0} = -1$ and so $y - 0 = -1(x - 6)$ or $y = -x + 6$.
For SB, slope = $\frac{0-4}{9-0} = -\frac{4}{9}$ and so $y - 0 = -\frac{4}{9}(x - 9)$ or $y = -\frac{4}{9}x + 4$.
So the point Y, which lies at the intersection QC and SB, we obtain by equating these lines

$$-x + 6 = -\frac{4}{9}x + 4$$
$$2 = \frac{5}{9}x$$
$$x = \frac{18}{5}$$

and so $y = -x + 6 = -\frac{18}{5} + 6 = \frac{12}{5}$ and thus Y is the point $\left(\frac{18}{5}, \frac{12}{5}\right)$.

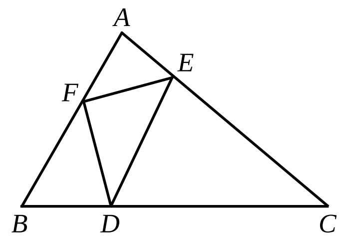
Now the line through $A(0, 0)$ and $X\left(\frac{9}{4}, \frac{3}{2}\right)$ has slope $m = \frac{\frac{3}{2} - 0}{\frac{9}{4} - 0} = \frac{2}{3}$ and so is $y = \frac{2}{3}x$.

The point Y lies on this line, as $\frac{12}{5}=\frac{2}{3}\left(\frac{18}{5}\right)$. [This could be done with L.S./R.S. format using equation of line.]
Therefore A, X, Y lie on the same line.

2. In ΔABC, the points D, E and F are on sides BC, CA and AB, respectively, such that $\angle AFE=\angle BFD$, $\angle BDF=\angle CDE$, and $\angle CED=\angle AEF$.

(a) Prove that $\angle BDF=\angle BAC$.

(b) If $AB=5$, $BC=8$ and $CA=7$, determine the length of BD.

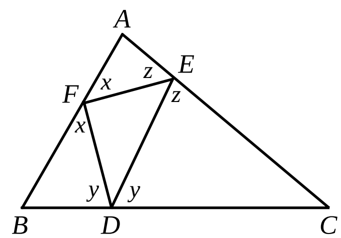


Solution

(a) Let $\angle AFE=\angle BFD=x$

$\angle BDF=\angle CDE=y$

$\angle CED=\angle AEF=z$



Thus $\angle FAE=180°-x-z$

$\angle FBD=180°-x-y$

$\angle ECD=180°-y-z$

and these 3 angles add to $180°$, so

$540°-2(x+y+z)=180°$

$x+y+z=180°$

Since $\angle FAE+\angle AFE+\angle AEF=180°$ (from ΔAEF)

$\angle FAE+x+z=x+y+z$

$\angle FAE=y$

Therefore $\angle BDF=\angle BAC$.

(b) Similarly to part (a), $\angle ECD=\angle BFD=x$, $\angle FBD=\angle CED=z$.

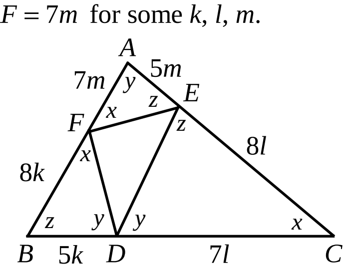
By equal angles, $\Delta ABC\sim\Delta DBF\sim\Delta DEC\sim\Delta AEF$ and so $\frac{BD}{BF}=\frac{BA}{BC}=\frac{5}{8}$, $\frac{CD}{CE}=\frac{CA}{CB}=\frac{7}{8}$, $\frac{AE}{AF}=\frac{AB}{AC}=\frac{5}{7}$.

Therefore, let $BD=5k$, $BF=8k$, $CD=7l$, $CE=8l$, $AE=5m$, $AF=7m$ for some k, l, m.

Then $5k+7l=8$ (1)

$7m+8k=5$ (2)

$5m+8l=7$ (3)



Determining 7 × (3) – 5 × (1) to eliminate m, we get

$56l-40k=49-25=24$

$7l-5k=3$ (4)

Calculating (1) – (4), we get $10k=5$ or $BD=5k=\frac{5}{2}$.

3. (a) Alphonse and Beryl are playing a game, starting with the geometric shape shown in Figure 1. Alphonse begins the game by cutting the original shape into two pieces along one of the lines. He then passes the piece containing the black triangle to Beryl, and discards the other piece.

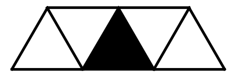


Figure 1

Beryl repeats these steps with the piece she receives; that is to say she cuts along the length of a line, passes the piece containing the black triangle back to Alphonse, and discards the other piece. This process continues, with the winner being the player who, at the beginning of his or her turn receives only the black triangle. Show, with justification, that there is always a winning strategy for Beryl.

Solution

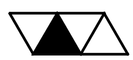
We first consider Alphonse's possible moves to begin the game. We can assume, without loss of generality, that he cuts on the left side of the black triangle.

Case 1

Alphonse removes two white triangles, leaving .

In this case, Beryl removes only one white triangle, and passes the shape  back to Alphonse, forcing him to remove the last white triangle and lose.

Case 2

Alphonse removes one white triangle only, leaving .

Beryl removes both of the white triangles on the right, leaving Alphonse in the same position as in Case 1 for his second turn.

Therefore Beryl can always win, regardless of Alphonse's strategy.

(b) Alphonse and Beryl now play a game with the same rules as in (a), except this time they use the shape in Figure 2 and Beryl goes first. As in (a), cuts may only be made along the whole length of a line in the figure. Is there a strategy that Beryl can use to be guaranteed that she will win? (Provide justification for your answer.)

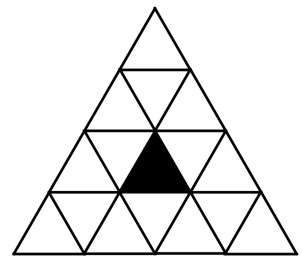


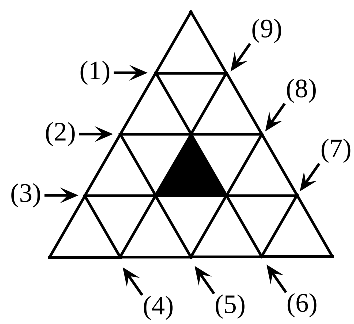
Figure 2

Solution

We show that, again, Beryl always has a winning strategy.

The strategy is to reduce the shape in Figure 2 to the shape in Figure 1, and to have Alphonse make the first cut at this stage. Beryl also knows that if she is forced into a position of being the first to cut when Figure 2 is reduced to Figure 1, then Alphonse can force her to lose.

We number the lines on the diagram for convenience.



We can assume without loss of generality (because of symmetry) that Beryl cuts along (1), (2) or (3) to begin.

If she cuts (2) or (3), then Alphonse cuts the other of these two and leaves Beryl with Figure 1, where she will lose.

Therefore Beryl cuts (1) to begin.

If Alphonse now cuts (2) or (3), Beryl cuts the other of these two and passes the shape in Figure 1 back to Alphonse, and so he loses.

If Alphonse cuts (8) or (9), Beryl cuts the opposite and passes the shape in Figure 1 to Alphonse, and so he loses. (Similarly, if he cuts (5) or (6)).

So assume that Alphonse cuts (4) or (7), say (4) by symmetry.

If Beryl now cuts any of (2), (3), (5), (6), (8), or (9), then Alphonse can force Beryl to lose, in the same way as she could have forced him to lose, as above. So Beryl cuts (7).

Now Alphonse is forced to cut one of (2), (3), (5), (6), (8), or (9), and so Beryl makes the appropriate cut, passing the shape in Figure 1 back to Alphonse, and so he must lose.

Therefore Beryl always can have a winning strategy.

4. A sequence $t_1, t_2, t_3, ..., t_n$ of n terms is defined as follows:

$$t_1 = 1,\ t_2 = 4,\ \text{and}\ t_k = t_{k-1} + t_{k-2}\ \text{for}\ k = 3, 4, ..., n.$$

Let T be the set of all terms in this sequence; that is, $T = \{t_1, t_2, t_3, ..., t_n\}$.

(a) How many positive integers can be expressed as the sum of exactly two distinct elements of the set T ?

Summary

Part (a) 4 ✓

Part (b) 6 ✓

Solution

$t_k > 0$ for all k, $1 \le k \le n$.

Also $t_k < t_{k+1}$ for all $k \le n-1$ since $t_{k+1} = t_k + t_{k-1}$.

Hence the sequence is monotone increasing.

The sum of any pair of terms is an integer and there are $\binom{n}{2}$ pairs.

Can any two pairs produce the same integer?

Consider $t_a + t_b$ and $t_c + t_d$. Clearly if $t_b = t_d$ then $t_a = t_c$ and vice versa, implying the same pair.

Hence none of the four terms is equal, so we can assume one term, say t_d to be the largest.

Then $t_d = t_{d-1} + t_{d-2} \ge t_a + t_b$, since the maximum values of t_a and t_b are t_{d-1} and t_{d-2} and they cannot be alike.

But since $t_c > 0$, $t_c + t_d > t_a + t_d$ and there are no two pairs that add to the same integer, so there are exactly $\binom{n}{2}$ integers possible.

(b) How many positive integers can be expressed as the sum of exactly three distinct elements of the set T ?

Solution

Consider $t_a + t_b + t_c$ and $t_d + t_e + t_f$. If any of the first three equals any of the second three we are left with pair sums of the remaining ones being equal, which is impossible from part (a). Hence there are six unlike terms, and again we can assume one, say t_f, to be the greatest.

It is clearly possible for equality by setting t_a and t_b equal to t_{f-1} and t_{f-2} and then t_d and t_e equal to t_{c-1} and t_{c-2}.

In how many ways can this be done for given *f*. Clearly, $6 \le f \le n$, and since $2 < c < f-2$, for any given *f* there are $f-5$ choices for *c* and the number of ways possible is $\sum_{f=6}^{n}(f-5) = 1+2+3+ \quad +(n-5) = \binom{n-4}{2}$.

There are a maximum of $\binom{n}{3} - \binom{n-4}{2}$ possible integers.

Of these, are any two of like sum?

In $t_a + t_b + t_c$, the maximal values are t_{f+1}, t_{f-3}, and t_{f-4}, since if one is t_{f-1} and one t_{f-2} we revert to the discussed state. Hence

$$\begin{aligned} t_a + t_b + t_c &\le t_{f-1} + t_{f-3} + t_{f-4} \\ &= t_{f-1} + t_{f-2} \\ &= t_f. \end{aligned}$$

But $t_d + t_e + t_f > t_f$.

Hence there are no other triples for which equality of sums exist, and the number of possible integers is $\binom{n}{3} - \binom{n-4}{2}$.