The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

The
Canadian Open Mathematics Challenge

Wednesday, November 29, 2000

Time: $2\frac{1}{2}$ hours

Calculators are NOT permitted.

Do not open this booklet until instructed to do so.
There are two parts to the paper.

PART A
This part of the paper consists of 8 questions, each worth 5 marks. You can earn full value for each question by entering the correct answer in the space provided. Any work you do in obtaining an answer will be considered for part marks if you do not have the correct answer, **provided that it is done in the space allocated** to that question in your answer booklet.

PART B
This part of the paper consists of 4 questions, each worth 10 marks. Finished solutions must be written in the appropriate location in the answer booklet. Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be provided by your supervising teacher. Any extra papers should be placed inside your answer booklet.
Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

Canadian Open Mathematics Challenge

NOTE: 1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. It is expected that all calculations and answers will be expressed as exact numbers such as $4\pi, 2+\sqrt{7}$, etc.
4. Calculators are **not** allowed.

PART A

1. An operation "Δ" is defined by $a \Delta b = 1 - \frac{a}{b}$, $b \neq 0$.

   What is the value of $(1 \Delta 2) \Delta (3 \Delta 4)$?

2. The sequence 9, 18, 27, 36, 45, 54, … consists of successive multiples of 9. This sequence is then altered by multiplying every other term by –1, starting with the first term, to produce the new sequence $-9, 18, -27, 36, -45, 54, \ldots$. If the sum of the first *n* terms of this new sequence is 180, determine *n*.

3. The symbol *n*! is used to represent the product $n(n-1)(n-2)\cdots(3)(2)(1)$.

   For example, $4! = 4(3)(2)(1)$. Determine *n* such that $n! = \left(2^{15}\right)\left(3^{6}\right)\left(5^{3}\right)\left(7^{2}\right)(11)(13)$.

4. The symbol $\lfloor x \rfloor$ means the greatest integer less than or equal to *x*. For example,

   $\lfloor 5.7 \rfloor = 5$, $\lfloor \pi \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.

   Calculate the value of the sum

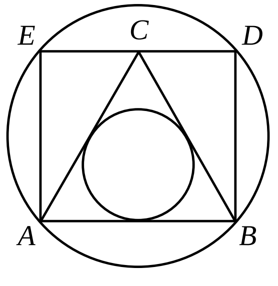
   $$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \lfloor \sqrt{4} \rfloor + \cdots + \lfloor \sqrt{48} \rfloor + \lfloor \sqrt{49} \rfloor + \lfloor \sqrt{50} \rfloor.$$

5. How many five-digit positive integers have the property that the product of their digits is 2000?

6. Solve the equation $4\left(16^{\sin^2 x}\right) = 2^{6 \sin x}$, for $0 \leq x \leq 2\pi$.

7. The sequence of numbers …, $a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3$, … is defined by $a_n - (n+1)a_{2-n} = (n+3)^2$, for all integers *n*. Calculate a_0.

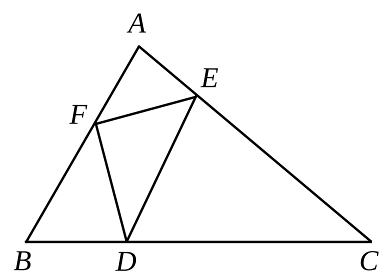
8. In the diagram, ΔABC is equilateral and the radius of its inscribed circle is 1. A larger circle is drawn through the vertices of the rectangle $ABDE$. What is the diameter of the larger circle?



PART B

1. Triangle ABC has vertices $A(0, 0)$, $B(9, 0)$ and $C(0, 6)$. The points P and Q lie on side AB such that $AP = PQ = QB$. Similarly, the points R and S lie on side AC so that $AR = RS = SC$.
The vertex C is joined to each of the points P and Q. In the same way, B is joined to R and S.
   (a) Determine the equation of the line through the points R and B.
   (b) Determine the equation of the line through the points P and C.
   (c) The line segments PC and RB intersect at X, and the line segments QC and SB intersect at Y. Prove that the points A, X and Y lie on the same straight line.

2. In ΔABC, the points D, E and F are on sides BC, CA and AB, respectively, such that $\angle AFE = \angle BFD$, $\angle BDF = \angle CDE$, and $\angle CED = \angle AEF$.
   (a) Prove that $\angle BDF = \angle BAC$.
   (b) If $AB = 5$, $BC = 8$ and $CA = 7$, determine the length of BD.



3. (a) Alphonse and Beryl are playing a game, starting with the geometric shape shown in Figure 1. Alphonse begins the game by cutting the original shape into two pieces along one of the lines. He then passes the piece containing the black triangle to Beryl, and discards the other piece.
Beryl repeats these steps with the piece she receives; that is to say, she cuts along the length of a line, passes the piece containing the black triangle back to Alphonse, and discards the other piece. This process continues, with the winner being the player who, at the beginning of his or her turn, receives only the black triangle. Show, with justification, that there is always a winning strategy for Beryl.

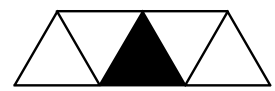


Figure 1

   (b) Alphonse and Beryl now play a game with the same rules as in (a), except this time they use the shape in Figure 2 and Beryl goes first. As in (a), cuts may only be made along the whole length of a line in the figure. Is there a strategy that Beryl can use to be guaranteed that she will win? (Provide justification for your answer.)

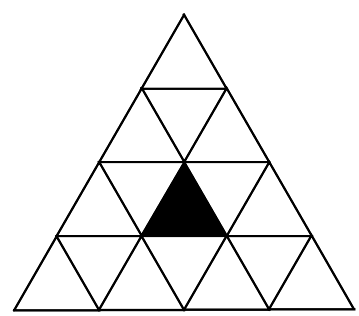


Figure 2

4. A sequence $t_1, t_2, t_3, ..., t_n$ of n terms is defined as follows:

$$t_1 = 1,\ t_2 = 4, \text{ and } t_k = t_{k-1} + t_{k-2} \text{ for } k = 3, 4, ..., n.$$

Let T be the set of all terms in this sequence; that is, $T = \{t_1, t_2, t_3, ..., t_n\}$.
   (a) How many positive integers can be expressed as the sum of exactly two distinct elements of the set T?
   (b) How many positive integers can be expressed as the sum of exactly three distinct elements of the set T?