The Canadian Mathematical Society
in collaboration with
The Center for Education
in Mathematics and Computing

The Third Canadian Open Mathematics Challenge

Wednesday, November 25, 1998

Solutions

Part A

Note: All questions in part A were graded out of 5 points.

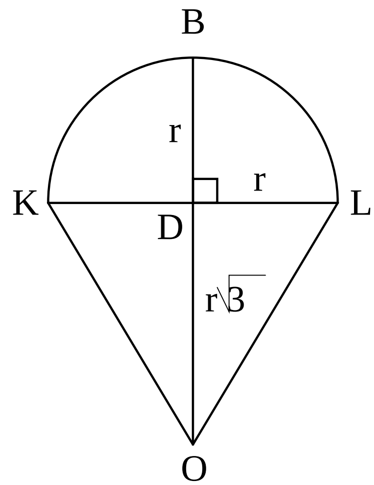
1. This question is most easily solved by bringing the 3^x to the left side, factoring and then arriving at, $3^x(8) = 216.$ This leads to the solution $x = 3$.

   The average was 4.0.

2. If we recognize that the area of opposite faces in the box are equal we arrive at the equations $2(2a^2)+2(2a)+2(2a) = 54$ or $2a^2+3a-27 = 0$. This leads to $a = 3$ with a volume of 18.

   The average was 4.3.

3. If we cut out part of the diagram and label appropriately we find that $DL = r$ and $DO = r\sqrt{3}$. This gives $OB = r(1+\sqrt{3}) = 6$ or $r = \dfrac{6}{1+\sqrt{3}}$.



The average was 1.7.

4. The easiest way to do this question is to recognize that the average of the 24 odd terms is $\frac{1272}{24} = 53$. This is also the average of all 47 terms. The sum of 47 terms is then $47 \times 53 = 2491$. This could also have been done with the use of formulae and a more standard approach.

The average was 1.8.

5. Using the fact that $log_a a^n = n log_a a = n$ we see that the numerical value of the series is $1 - 2 + 3 - 4 + \ldots - 98 + 99 = 50$.

The average was 3.2.

6. We first note that $DC = 60$ and $BD = 40$. If we use the cosine law in $\triangle ACD$ and the fact that $\cos C = \frac{3}{5}$, we find $AD = \sqrt{2880}$. We could also have done the problem by constructing a perpendicular D to AC and then solving similar triangles.

The average was 1.7.

7. There are numerous approaches to this problem each of which leads quickly to the answer. There are $\binom{10}{3}$ groups of three letters to be chosen. There are 5 ways to choose in A, 3 ways to choose a B and 2 ways to choose a C or $5 \times 3 \times 2$ ways of selecting an A, B or C. The required probability is $\dfrac{5 \times 3 \times 2}{\binom{10}{3}} = \dfrac{1}{4}$.

The average was 0.6.

8. Using basic properties of symmetry we observe that the corner of the box, the centre of the small sphere and the centre of the large sphere all lie along the same straight line drawn from the origin. The distance from the corner of the box to the centre of the small sphere is $r\sqrt{3}$. If the distance from the corner of the box to the centre of the large sphere is $16\sqrt{3}$ we can now write, $r\sqrt{3} + r + 15 = 16\sqrt{3}$ or $r = \frac{16\sqrt{3}-15}{\sqrt{3}+1}$.

   The average was 0.6.

Part B

Note: All questions in part B were graded out of 10 points.

1. This problem can be easily solved by first finding the coordinates of the vertices of $\triangle ABC$. The coordinates of P are then found by using the properties of right bisectors of the sides. Doing this, we find $P(6, 4)$ as the required point. The equation of the required line is then $x - 5y + 14 = 0$. ***Note:*** Students should first draw a diagram in doing this problem.

   The average was 5.0.

2. If we let DC be y and AD be x then we can represent YC as $\frac{6}{y}$ and $AX = \frac{10}{x}$. From this, $BY = \frac{xy-6}{y}$ and $BX = \frac{xy-10}{x}$ giving $\frac{xy-10}{x} \cdot \frac{xy-6}{y} = 8$ since $|\triangle BXY| = 4$. This leads to the solution $|\triangle DXY| = 2\sqrt{21}$.

   The average was 1.8.

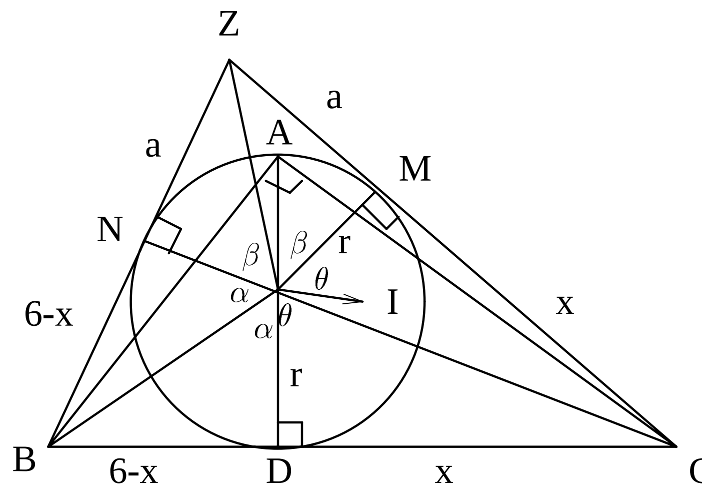
3.

(a) In essence, Alphonse adopts a strategy that will make Beryl enter the last ring first. This guarantees a win because in the final ring the moves are necessarily successive and there are an even number of regions thus guaranteeing Alphonse the 2nd, 4th, 6th and 8th position, i.e. the winning position

(b) Beryl adopts the strategy that will allow him to be the first to enter ring three and five. This guaranteese Beryl that he will always win because Alphonse will always be in an even position in these rings (if we label the regions 1,2,3,. . . ,9) when it is Beryl's turn to move.

The average was 2.7.

4. The best approach is to complete the diagram as shown and label appropriately. We now observe that $\alpha + \beta + \theta = 180°$ and using the double tangent formula we conclude that $\tan(\alpha+\beta) = -\tan\theta$ and then by the substitution $4r^2 = (6-x)(x)$ we can arrive at $a = 2$. It is easy to see that $ZB + ZC = 2 + (6-x) + 2 + x = 10$ as required. This problem can also be done using Heron's formula.



The average was 0.8.