The Third
Canadian Open
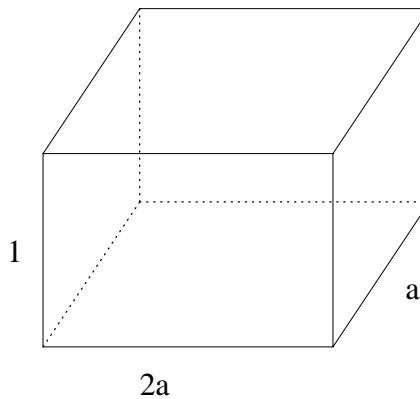# *Mathematics Challenge*

Wednesday, November 25, 1998

Examination Paper

Part A

Note: All questions in part A will be graded out of 5 points.

1. Solve for x, given that $3^{x+2} = 3^x + 216$.

2. A rectangular closed box (shown) with dimensions a, $2a$ and 1 has a surface area of 54, where a is an integer. Determine the volume of the box.

3. In the figure, each region T represents an equilateral triangle and each region S a semicircle. The complete figure is a semicircle of radius 6 with its centre O. The three smaller semicircles touch the large semicircle at points A, B and C. What is the radius of a semicircle S?
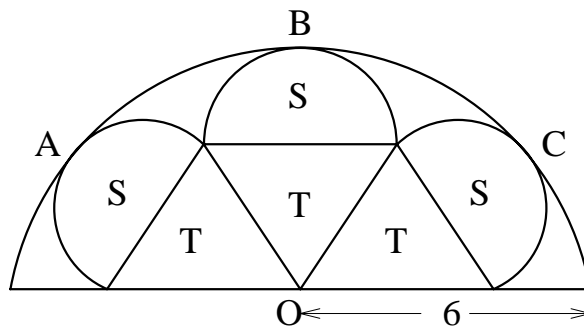
4. In an arithmetic sequence $t_1, t_2, t_3, ..., t_{47}$, the sum of the odd numbered terms is 1272. What is the sum of all 47 terms in the sequence?

5. Compute the sum of the first 99 terms of the series

$$\log_a a - log_a a^2 + log_a a^3 - log_a a^4 + log_a a^5 - log_a a^6 + \ldots$$

6. The lengths of the sides of triangle ABC are $60, 80$ and 100 with $\angle A = 90°$. The line AD divides triangle ABC into two triangles of equal perimeter. Calculate the length of AD.
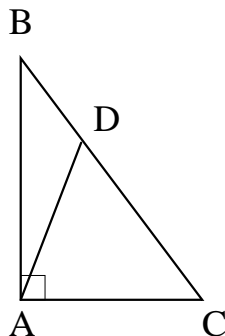
7. There are ten prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top ten contestants in a mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the eight contestant goes to select a prize, what is the probability that the remaining three prizes are one A, one B and one C?

8. Nine spheres are placed in a closed cubical box of side lenth 32 cm. Four small spheres of radius r are first placed in the bottom corners of the box so that they touch adjacent sides of the box but not each other. A large sphere of radius 15 cm is then placed in the box so that it touches each of the four smaller spheres but not the bottom. Four spheres of radius r are then added in the upper corners and the box closed so that the lid just touches the four smaller spheres. Calculate r.

Part B
Note: Answer all questions. The problems in this section are worth 10 marks each. Marks will be based on presentation. A correct solution poorly presented will not earn full marks.

1. Triangle ABC has its sides determinded in the following way: side AB by line $3x - 2y + 3 = 0$; side BC by line $x + y - 14 = 0$; and side AC by line $y = 3$. If the point P is chosen so that $PA = PB = PC$, determine the equation of the line containing A and P.

2. $ABCD$ is a rectangle and lines DX, DY and XY are drawn where X is on AB and Y is on BC. The area of triangle AXD is 5, the area of triangle BXY is 4 and the area of triangle CYD is 3. Determine the area of triangle DXY.
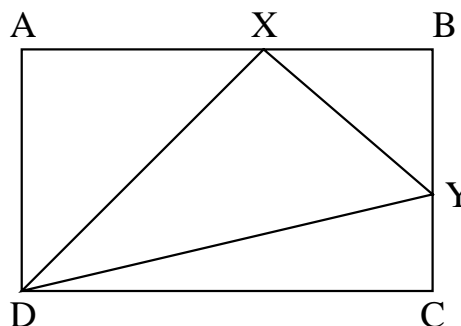
3. Alphonse and Beryl play a game by alternately moving a disk on a circular board. The game starts with the disk already on the board as shown. A player may move either clockwise one position or one position toward the centre but cannot move to a position that has been previously occupied. The last person who is able to move wins the game.
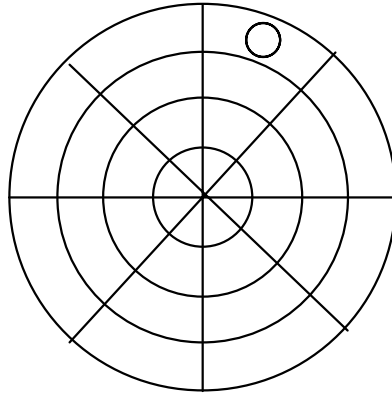
(a) If Alphonse moves first, is there a strategy which guarantees that he will always win?

(b) Is there a winning strategy for either of the players if the board is changed to five concentric circles with nine regions in each ring and Alphonse moves first? (The rules for playing this new game remain the same)

4. A line segment BC has length 6. Point A is chosen such that $\angle BAC$ is a right angle. For any position of A a point D is chosen in BC so that AD is perpendicular to BC. A circle with AD as diameter has tangents drawn from C and B to touch the circle at M and N, respectively, with these tangents intersecting at Z. Prove that $ZB + ZC$ is constant.