### The Canadian Mathematical Society

in collaboration with

## The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

# First Canadian Open Mathematics Challenge (1996)
# Solutions

Note: All questions in Part A were graded out of a possible 5 points.

**Part A**

1. The roots of the equation $x^2 + 4x - 5 = 0$ are also the roots of the equation

   $2x^3 + 9x^2 - 6x - 5 = 0$ .

   What is the third root of the second equation?

   The third root can be found either by dividing the quadratic expression into the cubic expression or by factoring the two expressions. Many students, having done the division, did not determine the third root, which is $-\frac{1}{2}$.
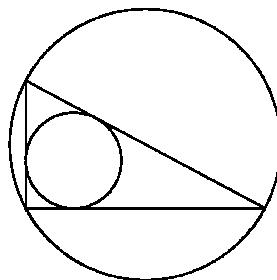
   The average score was 3.9.

2. The numbers $a$, $b$, $c$ are the digits of a three digit number which satisfy $49a + 7b + c = 286$. What is the three digit number $(190a + 10b + c)$ ?

   This problem can be handled by guessing and verifying the guess. It is also nicely solved by using the fact that 7 divides $49a + 7b$, so the remainder on the left side is $c$, while on the right side division by 7 gives remainder 6; hence $c$ is 6. Continuing in the same way, $a = b = 5$, and the answer is 556.

   The average score was 3.9.

3. The vertices of a right angled triangle are on a circle of radius $R$ and the sides of the triangle are tangent to another circle of radius $r$. If the lengths of the sides about the right angle are 16 and 30, determine the value of $R + r$.

Since the triangle is right–angled, the hypotenuse is the diameter of the circle, which gives $R = 17$. By using the area of the whole triangle and the sum of the three small triangles having height $r$, $r$ is determined to be 6. Hence $R + r = 23$.

The average score was 2.1

4. Determine the smallest positive integer, $n$, which satisfies the equation $n^3 + 2n = b$, where $b$ is the square of an odd integer.

This question is solved easily by observing that $n^3 + 2n^2 = n^2(n + 2)$. Since $n^2$ is a square, so too is $n + 2$ and 7 is the smallest value that makes $n + 2$ an odd square.

The average score was 2.3

5. A road map of Grid City is shown in the diagram. The perimeter of the park is a road but there is no road through the park. How many different shortest road routes are there from point $A$ to point $B$?
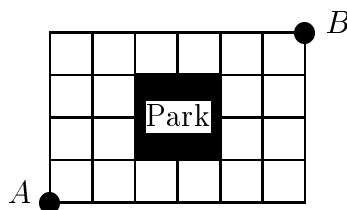
The solution to this problem depends on two observations. First, there are identical numbers of routes going above the park or below the park; second, going above the park requires going through either $X$ or $Y$ and one cannot go through both.
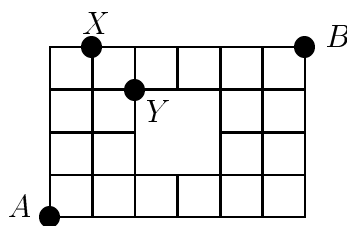
There are 5 routes from $A$ to $X$ and for each there is only one way of proceeding to $B$. There are 10 routes from $A$ to $Y$ and for each there are 5 routes to $B$. Hence there are 110 routes in total.
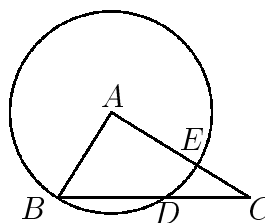
The average score was 0.9.

6. In a 14 team baseball league, each team played each of the other teams 10 times. At the end of the season, the number of games won by each team differed from those won by the team that immediately followed it by the same amount. Determine the greatest number of games the last place team could have won, assuming that no ties were allowed.

Each team plays 130 games, so there are 910 games in all. Using an arithmetic sequence with difference 2 (it cannot be 1) gives a first term of 52, the number of games won by the last team.

The average score was 1.7.

7. Triangle $ABC$ is right angled at $A$. The circle with center $A$ and radius $AB$ cuts $BC$ and $AC$ internally at $D$ and $E$ respectively. If $BD = 20$ and $DC = 16$, determine $AC^2$.



This Problem is done most easily by dropping a perpendicular from $A$ to $BC$ and using the properties of similar triangles. The answer is 936.

The average score was 0.7.

8. Determine all pairs of integers $(x, y)$ which satisfy the equation

$6x^2 - 3xy - 13x + 5y = -11.$

Any question of this type is handled by solving for one variable in terms of the other. This leads to a fraction in which the denominator is an expression that must divide the numerator if integers are to result. Here, solving for $y$ in terms of $x$ forces $x$ to have values 2 or 1, resulting in $y$ being 9 or $-2$.

The average score was 1.2.

9. If $\log_{2n}(1944) = \log_n \left(486\sqrt{2}\right)$, compute $n^6$.

Whenever in doubt with logarithms, go to fundamentals. If $\log_b a = c$, then $b^c = a$. Using this with each of the given expressions leads to an answer of $3^{20} \cdot 2^6$.

The average score was 1.0.

10. Determine the sum of the angles $A, B$, where $0° \leq A, B \leq 180°$ and $\sin A + \sin B = \sqrt{\frac{3}{2}}$, $\cos A + \cos B = \sqrt{\frac{1}{2}}$.

The most direct approach is to square both equations and add. From this one determines that $\cos(A - B) = 0$, and $A - B = \pm 90°$. By considering each possibility, one determines that $A + B = 120°$.

The average score was 0.6.

## Part B

1. Three numbers form an arithmetic sequence, the common difference being 11. If the first number is decreased by 6, the second is decreased by 1 and the third is doubled, the resulting numbers are in geometric sequence. Determine the numbers which form the arithmetic sequence.

   The given conditions lead to three numbers that must form a geometric sequence. Using these and the fact that the ratio of consecutive terms is constant leads to a quadratic equation which has two solutions and hence two possible arithmetic sequences $14, 25, 36$, or $-26, -15, -4$.

   The average score was 3.6.

2. A rectangle $ABCD$ has diagonal of length $d$. The line $AE$ is drawn perpendicular to the diagonal $BD$. The sides of the rectangle $EFCG$ have lengths $n$ and 1. Prove $d^{2/3} = n^{2/3} + 1$.
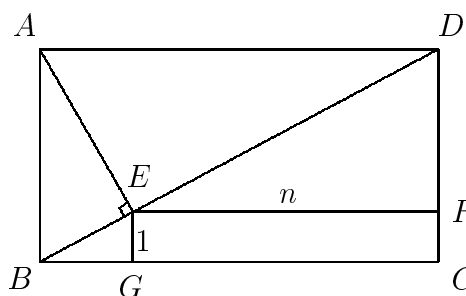


   There are many ways of getting at this problem, including analytic geometry. In all cases, one must introduce variables. If one sets $BG = x$, only one variable is required. Using similar triangles and the Pythagorean Theorem, $BE = \sqrt{x^2 + 1}$, $AB = x^2 + 1$, $DF = x^2$ and $AD = x + n$, which leads to $n = x^3$. The desired result then follows from $BD^2 = AD^2 + AB^2$.

3. (a) Given positive numbers $a_1, a_2, a_3, \ldots, a_n$ and the quadratic function

$$f(x) = \sum_{i=1}^{n} (x - a_i)^2 \ ,$$

   show that $f(x)$ attains its minimum value at $\dfrac{1}{n} \sum_{i=1}^{n} a_i$, and prove that

$$\sum_{i=1}^{n} a_i^{\ 2} \geq \frac{1}{n} \left( \sum_{i=1}^{n} a_i \right)^2 \ .$$

**(b)** The sum of sixteen positive numbers is 100 and the sum of their squares is 1000. Prove that none of the sixteen numbers is greater than 25.

The first part of the solution requires only the completing of the square on the given function and noting that the function is always equal to or greater than 0. The second part requires that one use the result from that of part (a) together with the greatest of the numbers expressed in terms of the remaining fifteen numbers and similarly with the squares of the numbers.