# *SOLUTIONS*

## QUESTION 1

**Solution**

**Note that**

$$f(1-x) = \frac{9^{1-x}}{9^{1-x}+3} = \frac{9}{9+3\times 9^{x}} = \frac{3}{9^{x}+3},$$

**from which we get**

$$f(x) + f(1-x) = \frac{9^{x}}{9^{x}+3} + \frac{3}{9^{x}+3} = 1.$$

**Therefore,**

$$\begin{aligned}
&\sum_{k=1}^{1995} f\left(\frac{k}{1996}\right) \\
&= \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right) + f\left(\frac{1996-k}{1996}\right)\right] + f\left(\frac{998}{1996}\right) && (1)\\
&= \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right) + f\left(1-\frac{k}{1996}\right)\right] + f\left(\frac{1}{2}\right) && (2)\\
&= 997 + \frac{3}{3+3} = 997\frac{1}{2}. && (3)
\end{aligned}$$

## QUESTION 2

**Solution 1.**

We prove equivalently that $a^{3a}b^{3b}c^{3c} \geq (abc)^{a+b+c}$. Due to complete symmetry in $a$, $b$ and $c$, we may assume, without loss of generality, that $a \geq b \geq c$. Then $a-b \geq 0$, $b-c \geq 0$, $a-c \geq 0$ and $\frac{a}{b} \geq 1$, $\frac{b}{c} \geq 1$, $\frac{a}{c} \geq 1$. Therefore,

$$\frac{a^{3a}b^{3b}c^{3c}}{(abc)^{a+b+c}} = \left(\frac{a}{b}\right)^{a-b}\left(\frac{b}{c}\right)^{b-c}\left(\frac{a}{c}\right)^{a-c} \geq 1.$$

**Solution 2.**

If we assign the weights $a, b, c$ to the numbers $a, b, c$, respectively, then by the wighted geometric-mean-harmonic-mean inequality followed by the arithmetic-mean-geometric-mean inequality, we get

$$\sqrt[a+b+c]{a^a b^b c^c} \geq \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}} = \frac{a+b+c}{3} \geq \sqrt[3]{abc},$$

from which $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ follows immediately.

## QUESTION 3

**Solution**

For convenience, the interior angle in a boomerang which is greater than $180^\circ$ will be called a "reflex angle".

Clearly, there are $b$ reflex angles, each occurring in a different boomerang and each with the corresponding vertex in the interior of $C$. Angles around these vertices add up to $2b\pi$. On the other hand, the sum of all the interior angles of $C$ is $(s-2)\pi$ and the sum of the interior angles of all the $q$ quadrilaterals is $2\pi q$.

Therefore, $2\pi q \geq 2b\pi + (s-2)\pi$ from which $q \geq b + \frac{s-2}{2}$ follows.

## QUESTION 4

**Solution 1.**

**Since $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$, we see that when $k = 0$, $(x_1, x_2, \cdots, x_n; y) = \left(1, 2, \cdots, n; \frac{n(n+1)}{2}\right)$ is a solution. To see that we can generate infinitely many solutions in general, set $c = \frac{n(n+1)}{2}$ and notice that for all positive integers $q$, we have:**

$$\begin{aligned}
&(c^k q^{3k+2})^3 + (2c^k q^{3k+2})^3 + \cdots + (nc^k q^{3k+2})^3 \\
&\quad = c^{3k} q^{3(3k+2)}(1^3 + 2^3 + \cdots n^3) \\
&\quad = c^{3k} q^{3(3k+2)} \left(\frac{n(n+1)}{2}\right)^2 \\
&\quad = c^{3k+2} q^{3(3k+2)} = (cq^3)^{3k+2}.
\end{aligned}$$

**That is, $(x_1, x_2, \cdots, x_n; y) = (c^k q^{3k+2}, 2c^k q^{3k+2}, \cdots, nc^k q^{3k+2}; cq^3)$ is a solution. This completes the proof.**

**Solution 2.**

**For any positive integer $q$, take $x_1 = x_2 = \cdots = x_n = n^{2k+1} q^{3k+2}$, $y = n^2 q^3$. Then**

$$\sum_{i=1}^{n} x_i^3 = n \cdot n^{6k+3} \cdot q^{9k+6} = (n^2 q^3)^{3k+2} = y^{3k+2}.$$

**Solution 3.**

**If** $n = 1$, **take** $x_1 = q^{3k+2}, y = q^3$ **as in solution 2. For** $n > 1$, **we look for solutions of the form**

$$x_1 = x_2 = \cdots x_n = n^p, \ \ y = n^q.$$

**Then**

$$\sum_{i=1}^{n} x_i^3 = y^{3k+2} \Leftrightarrow n^{3p+1} = n^{(3k+2)q} \Leftrightarrow 3p + 1 = (3k + 2)q \Leftrightarrow (3k + 2)q - 3p = 1.$$

**The last equation is satisfied if we take**

$$q = 3t + 2 \text{ and } p = (3k + 2)t + (2k + 1) \text{ where } t$$

**is any nonnegative integer. Thus, infinetely many solutions in positive integers are given by**

$$x_1 = x_2 = \cdots x_n = n^{(3k+2)t+(2k+1)}, \ \ y = n^{3t+2}.$$

## QUESTION 5

**Solution**

**Note first that** $u_1 = 1 - u$. **Since for all** $x \in [u, 1]$, $u \le x$ **and** $1 - x \le 1 - u$ **we have**

$$\begin{aligned}
1 - &\left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2 \\
&= 1 - ux - (1-u)(1-x) - 2\sqrt{ux(1-u)(1-x)} \\
&= u + x - 2ux - 2\sqrt{ux(1-u)(1-x)} \\
&\le u + x - 2ux - 2u(1-x) = x - u.
\end{aligned}$$

**Therefore,**

$$f(x) = 0 \text{ if } 0 \le x \le u \tag{1}$$

**and**

$$f(x) \le x - u \text{ if } u \le x \le 1. \tag{2}$$

**From (2) we get** $u_2 = f(u_1) \le u_1 - u = 1 - 2u$ **if** $u_1 \ge u$. **An easy induction then yields**

$$u_{n+1} = f(u_n) \le u_n - u \le 1 - (n+1)u \text{ if } u_i \ge u \text{ for all } i = 1, 2, \cdots, n.$$

**Thus for sufficiently large** $k$, **we must have** $u_{k-1} < u$ **and then** $u_k = f(u_{k-1}) = 0$ **by (1).**