43$^{\text{rd}}$ Canadian Mathematical Olympiad

Wednesday, March 23, 2011

CMS
SMC

(1) Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9, and 0 do not appear). Show that no number of this form can divide another number of this form.

(2) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

(3) Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

(4) Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a that is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

(5) Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where for any k, $\epsilon_k = 1$ or $\epsilon_k = -1$, such that
$$S = \epsilon_1(1 + d)^2 + \epsilon_2(1 + 2d)^2 + \epsilon_3(1 + 3d)^2 + \cdots + \epsilon_n(1 + nd)^2.$$