Canadian Mathematical Olympiad
1984

Problem 1

Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

Problem 2

Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes place:

Alice: Are you going to cover your keys?

Bob: I would like to, but there are only 7 colours and I have 8 keys.

Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

Bob: You must be careful what you mean by "next to" or "three keys over from" since you can turn the key ring over and the keys are arranged in a circle.

Alice: Even so, you don't need 8 colours.

Problem: What is the smallest number of colours needed to distinguish n keys if all the keys are to be covered.

Problem 3

An integer is *digitally divisible* if
(a) none of its digits is zero;
(b) it is divisible by the sum of its digits (*e.g.*, 322 is digitally divisible).
Show that there are infinitely many digitally divisible integers.

Problem 4

An acute-angled triangle has unit area. Show that there is a point inside the triangle whose distance from each of the vertices is at least $\frac{2}{\sqrt[4]{27}}$.

Problem 5

Given any 7 real numbers, prove that there are two of them, say x and y, such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$