Canadian Mathematical Olympiad
1972

PROBLEM 1

Given three distinct unit circles, each of which is tangent to the other two, find the radii of the circles which are tangent to all three circles.

PROBLEM 2

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Define M to be the sum of all products of pairs $a_i a_j$ $(i < j)$, *i.e.*,

$$M = a_1(a_2 + a_3 + \cdots + a_n) + a_2(a_3 + a_4 + \cdots + a_n) + \cdots + a_{n-1}a_n.$$

Prove that the square of at least one of the numbers $a_1, a_2, \ldots, a_n$ does not exceed $2M/n(n-1)$.

PROBLEM 3

a) Prove that 10201 is composite in any base greater than 2.
b) Prove that 10101 is composite in any base.

PROBLEM 4

Describe a construction of a quadrilateral $ABCD$ given:
(i) the lengths of all four sides;
(ii) that AB and CD are parallel;
(iii) that BC and DA do not intersect.

PROBLEM 5

Prove that the equation $x^3 + 11^3 = y^3$ has no solution in positive integers x and y.

PROBLEM 6

Let a and b be distinct real numbers. Prove that there exist integers m and n such that $am + bn < 0$, $bm + an > 0$.

PROBLEM 7

a) Prove that the values of x for which $x = (x^2 + 1)/198$ lie between $1/198$ and $197.99494949\cdots$.
b) Use the result of a) to prove that $\sqrt{2} < 1.41\overline{421356}$.
c) Is it true that $\sqrt{2} < 1.41421356$?

PROBLEM 8

During a certain election campaign, p different kinds of promises are made by the various political parties $(p > 0)$. While several parties may make the same promise, any two parties have at least one promise in common; no two parties have exactly the same set of promises. Prove that there are no more than 2^{p-1} parties.

PROBLEM 9

Four distinct lines L_1, L_2, L_3, L_4 are given in the plane: L_1 and L_2 are respectively parallel to L_3 and L_4. Find the locus of a point moving so that the sum of its perpendicular distances from the four lines is constant.

PROBLEM 10

What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose entries all come from the set of integers between 100 and 1000 inclusive?