Mathematical Olympiads' Correspondence Program - 1997-1998

Problem Set 8

43. The sides BC, CA, AB of triangle ABC are produced to the poins R, P, Q respectively, so that $CR = AP = BQ$. Prove that triangle PQR is equilateral if and only if triangle ABC is equilateral.

44. Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y^2+2y = x^3-x^2-x$ is satisfied by $(x,y) = (a(t)/c(t), b(t)/c(t))$.

45. ABC is a triangle with circumcentre O such that $\angle A$ exceeds $90°$ and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN and AC. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

46. Determine all functions f from the set of reals to the set of reals which satisfy the functional equation

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$$

for all real x and y.

47. Let x, y and z be positive real numbers. Show that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1 \ .$$

48. For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a}\times(\mathbf{b}-\mathbf{c})]^2 + [\mathbf{b}\times(\mathbf{c}-\mathbf{a})]^2 + [\mathbf{c}\times(\mathbf{a}-\mathbf{b})]^2 = (\mathbf{b}\times\mathbf{c})^2 + (\mathbf{c}\times\mathbf{a})^2 + (\mathbf{a}\times\mathbf{b})^2 + (\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})^2 \ .$$

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

43.

43. *First solution.* Suppose that triangle ABC is equilateral. A rotation of $60°$ about the centroid of ΔABC will rotate the points R, P and Q. Hence ΔPQR is equilateral. On the other hand, suppose, wolog, that $a \geq b \geq c$, with $a > c$. Then, for the internal angles of ΔABC, $A \geq B \geq C$. Suppose that $|PQ| = r$, $|QR| = p$ and $|PR| = q$, while s is the common length of the extensions. Then

$$p^2 = s^2 + (a+s)^2 + 2s(a+s)\cos B$$

and

$$r^2 = s^2 + (c+s)^2 + 2s(c+s)\cos A \ .$$

Since $a > c$ and $\cos B \geq \cos A$, we find that $p > r$, and so ΔPQR is not equilateral.

44.

44. *First solution.* The equation can be rewritten $(y+1)^2 = (x-1)^2(x+1)$. Let $x+1 = t^2$ so that $y+1 = (t^2-2)t$. Thus, we obtain the solution

$$(x,y) = (t^2-1, t^3-2t-1) \ .$$

With these polynomials, both sides of the equation are equal to $t^6 - 4t^4 + 4t^2 - 1$.

45.

45. *First solution.* Assign coordinates: $A \sim (0,0)$, $B \sim (2\cos\theta, 2\sin\theta)$, $C \sim (2u, 0)$ where $90° < \theta < 180°$ and $u > 1$. First, we determine O as the intersection of the right bisectors of AB and AC. The centre of AB has coordinates $(\cos\theta, \sin\theta)$ and its right bisector has equation

$$(\cos\theta)x + (\sin\theta)y = 1 \ .$$

The centre of segment AC has coordinates $(u, 0)$ and its right bisector has equation $x = u$. Hence, we find that

$$O \sim \left(u, \frac{1 - u\cos\theta}{\sin\theta}\right)$$

$$N \sim \left(\frac{1}{2}u, \frac{1 - u\cos\theta}{2\sin\theta}\right)$$

$$M \sim (u + \cos\theta, \sin\theta)$$

and

$$D \sim (u + 1, 0) \ .$$

The slope of MD is $(\sin\theta)/(\cos\theta - 1)$. The slope of ND is $(u\cos\theta - 1)/((u + 2)\sin\theta)$. Equating these two leads to the equation

$$u(\cos^2\theta - \sin^2\theta - \cos\theta) = 2\sin^2\theta + \cos\theta - 1$$

which reduces to

$$(u + 1)(2\cos^2\theta - \cos\theta - 1) = 0 \ .$$

Since $u + 1 > 0$, we have that $0 = 2\cos^2\theta - \cos\theta - 1 = (2\cos\theta + 1)(\cos\theta - 1)$. Hence $\cos\theta = -1/2$ and so $\angle A = 120°$.

46.

46. *First solution.* Let u and v be any pair of real numbers. We can solve $x + y = u$ and $x - y = v$ to obtain

$$(x, y) = \left(\frac{1}{2}(u + v), \frac{1}{2}(u - v)\right) \ .$$

From the functional equation, we find that $vf(u) - uf(v) = (u^2 - v^2)uv$, whence

$$\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2 \ .$$

Thus $(f(x)/x) - x^2$ must be some constant a, so that $f(x) = x^3 + ax$. This checks out for any constant a.

47.

47. *First solution.*

$$\begin{aligned}(x - \sqrt{yz})^2 \geq 0 &\Longrightarrow x^2 + yz \geq 2x\sqrt{yz} \\ &\Longrightarrow (x + y)(x + z) = x^2 + x(y + z) + yz \geq x(y + 2\sqrt{yz} + z) = x(\sqrt{y} + \sqrt{z})^2 \ .\end{aligned}$$

Hence

$$\frac{x}{x + \sqrt{(x + y)(x + z)}} \leq \frac{x}{x + \sqrt{x}(\sqrt{y} + \sqrt{z})} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \ .$$

Similarly

$$\frac{y}{y+\sqrt{(y+z)(y+x)}} \leq \frac{\sqrt{y}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

and

$$\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq \frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \quad .$$

Adding these inequalities yields the result.

48.

48. *First solution.* Let $\mathbf{u} = \mathbf{b} \times \mathbf{c}$, $\mathbf{v} = \mathbf{c} \times \mathbf{a}$ and $\mathbf{w} = \mathbf{a} \times \mathbf{b}$. Then, for example, $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = \mathbf{v} + \mathbf{w}$. The left side is equal to

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})+(\mathbf{u}+\mathbf{w})\cdot(\mathbf{u}+\mathbf{w})+(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = 2[(\mathbf{u}\cdot\mathbf{u})+(\mathbf{v}\cdot\mathbf{v})+(\mathbf{w}\cdot\mathbf{w})+(\mathbf{u}\cdot\mathbf{v})+(\mathbf{v}\cdot\mathbf{w})+(\mathbf{w}\cdot\mathbf{u})]$$

while the right side is equal to

$$(\mathbf{u}\cdot\mathbf{u}) + (\mathbf{v}\cdot\mathbf{v}) + (\mathbf{w}\cdot\mathbf{w}) + (\mathbf{u}+\mathbf{v}+\mathbf{w})^2$$

which expands to the final expression for the left side.

48. *Second solution.* For vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we have the identities

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

and

$$\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})\cdot\mathbf{w} \ .$$

Using these, we find for example that

$$\begin{aligned}
[\mathbf{a} \times (\mathbf{b}-\mathbf{c})]\cdot[\mathbf{a} \times (\mathbf{b}-\mathbf{c})] &= [\mathbf{a} \times (\mathbf{b}-\mathbf{c}) \times \mathbf{a}]\cdot(\mathbf{b}-\mathbf{c}) \\
&= \{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}-\mathbf{c}) - [(\mathbf{b}-\mathbf{c})\cdot\mathbf{a}]\mathbf{a}\}\cdot(\mathbf{b}-\mathbf{c}) \\
&= |\mathbf{a}|^2[|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b}\cdot\mathbf{c})] - [(\mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{a}]^2 \\
&= |\mathbf{a}|^2[|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b}\cdot\mathbf{c})] - (\mathbf{b}\cdot\mathbf{a})^2 - (\mathbf{c}\cdot\mathbf{a})^2 + 2(\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{a}) \quad .
\end{aligned}$$

Also

$$\begin{aligned}
(\mathbf{b} \times \mathbf{c})\cdot(\mathbf{b} \times \mathbf{c}) &= [(\mathbf{b}\cdot\mathbf{b})\mathbf{c} - (\mathbf{c}\cdot\mathbf{b})\mathbf{b}]\cdot\mathbf{c} \\
&= |\mathbf{b}|^2|\mathbf{c}|^2 - (\mathbf{c}\cdot\mathbf{b})^2
\end{aligned}$$

and

$$(\mathbf{b} \times \mathbf{c})\cdot(\mathbf{c} \times \mathbf{a}) = [(\mathbf{b} \times \mathbf{c}) \times \mathbf{c}]\cdot\mathbf{a} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a}) - (\mathbf{c}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{a}) \ .$$

From these the identity can be checked.