Mathematical Olympiads' Correspondence Program - 1997-1998

Problem Set 7

37. Let $f(x) = x^3 + ax^2 + bx + b$. Determine all integer pairs (a, b) for which $f(x)$ is the product of three linear factors with integer coefficients.

38. Assume that a face S of a convex polyhedron $\mathcal{P}$ has a common edge with every other face of $\mathcal{P}$. Show that there exists a simple (nonintersecting) closed (not necessarily planar) polygon that consists of edges of $\mathcal{P}$ and passes through all the vertices.

39. Let A_n be the set of mappings $f : \{1, 2, 3, \cdots, n\} \longrightarrow \{1, 2, 3, \cdots, n\}$ such that, if $f(k) = i$ for some i, then f also assumes all the values $1, 2, \cdots, i-1$. Prove that the number of elements of A_n is $\sum_{k=0}^{\infty} k^n 2^{-(k+1)}$.

40. Let x, y be positive real numbers. Find all solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} \ .$$

(*Note:* This is Problem 2268 in **CM+MM**, October, 1997.)

41. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7. (*Note:* This is Problem 2273 in $CM + MM$, October, 1997.)

42. ABC is an isosceles triangle with $AB = AC$. Let D be the point on the side AC for which $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90°$. Prove that $\angle ABP = \angle PCB$. (*Note:* This is Problem 2254 in $CM + MM$, September, 1997.)

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

37.

37. *First solution.* If $b = 0$, then the polynomial becomes $x^2(x+a)$, which satisfies the condition for all values of a. This covers the situation for which x is a factor of the polynomial. Since the leading coefficient of $f(x)$ is 1, the same must be true (up to sign) of its factors. Assume that $f(x) = (x+u)(x+v)(x+w)$ for integers u, v and w with $uvw \neq 0$. Since $uvw = uv + vw + wu = b$,

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 1 \ .$$

It is clearly not possible for all of u, v and w to be negative. Nor can it occur that two of them, say v and w can be negative, for then the left side would be less than $1/u \leq 1$. Suppose that u and v are positive, while w is negative. One possibility is that $u = 1$ and $v = -w$ in which case $f(x) = (x+1)(x^2 - v^2) = x^3 + x^2 - v^2x - v^2$. If neither u nor v is equal to 1, then $1/u + 1/v + 1/w < 1/u + 1/v \leq 1$, and this case is not possible. Finally, suppose that u, v and w are all positive, with $u \leq v \leq w$. Then $1 \leq 3/u$, so that $u \leq 3$. A little trial and error leads to the possibilities $(u, v, w) = (3, 3, 3), (2, 4, 4)$ and $(2, 3, 6)$. Thus the possibilities for (a, b) are $(u, 0)$, $(1, -v^2)$, $(9, 27)$, $(10, 32)$ and $(11, 36)$. Indeed, $x^3 + 9x^2 + 27x + 27 = (x+3)^3$, $x^3 + 10x^2 + 32x + 32 = (x+2)(x+4)^2$ and $x^3 + 11x^2 + 36x + 36 = (x+2)(x+3)(x+6)$.

38.

38. *First solution.* Suppose that the face S has m vertices A_1, A_2, $\cdots$, A_m listed in order, and that there are n vertices of $\mathcal{P}$ not contained in S. We prove the result by induction on n. If $n = 1$, then every

face abutting S is a triangle. Let X be the vertex off S; then $A_1 \cdots A_m X A_1$ is a polygonal path of the desired type. Suppose that the result holds for any number of vertices m of S and for n vertices off S where $1 \leq n \leq k$. Consider the case $n = k + 1$.

Consider the graph G of all vertices of $\mathcal{P}$ and those edges of $\mathcal{P}$ not bounding S. Since there are no faces bounded solely by these edges, the graph must be a *tree* (*i.e.*, it contains no loops and there is a unique path joining any pair of points). We show that there is at least one vertex X not in S for which every edge but one must connect X to a vertex of S. Suppose otherwise. Then, let us start with such a vertex X and form a sequence $X_1, X_2, \cdots$ of vertices not in S such that $X_i X_{i+1}$ are edges of $\mathcal{P}$. Since the number of vertices off S is finite, there must be $i < j$ for which $X_i = X_j$ so that $X_i X_{i+1} \cdots X_{j-1} X_j$ is a loop in G. But this contradicts the fact that G is a tree.

Hence there is a vertex X with at most one adjacent edge not connecting it to S. If there were no such edge, then X would be the only vertex not in S, contradicting $k + 1 \geq 2$. Hence there is a vertex Y not in S such that XY is an edge of $\mathcal{P}$. We may assume that Y is further from the plane of S than S. (If not, suppose that S is in the plane $z = 0$ and that $\mathcal{P}$ lies in the quadrant $z > 0$, $y > 0$ with Y further than X from the plane $y = 0$. We can transform $\mathcal{P}$ by a mapping of the type $(x, y, z) \rightarrow (x, y, z + \lambda y)$ for suitable positive λ. This will not alter the configuration of vertices and edges.) Extend YX to a point Z in the plane of S. Let $\mathcal{Q}$ be the convex hull of (smallest closed convex set containing) Z and $\mathcal{P}$. This will have a side T containing S of the form $A_1 A_2 \cdots A_r Z A_s \cdots A_m$ where $r < s$. The triangles XZA_r and XZA_s will be coplanar with faces of $\mathcal{P}$, and the convex hull will have at most k vertices not on T. Every face of $\mathcal{Q}$ will abut T. By the induction hypothesis, we can construct a polygon containing each vertex of $\mathcal{Q}$. If an edge of this polygon is YZ and so includes X, and if one edge is say ZA_r, then we can replace these two edges by $YXA_s A_{s-1} \cdots A_{r+1} A_r$. If YZ is not an edge of this polygon, but $A_r Z$ and ZA_s are, then we can replace these edges by $A_r X A_{r+1} \cdots A_s$. In both cases, we obtain a polygon of the required type for $\mathcal{P}$.

39.

39. *First solution.* Let $u_0 = 1$ and, for $n \geq 1$, let u_n be the number of elements in A_n. Let $1 \leq r \leq n$. Consider the set of mappings in A_n for which the value 1 is assumed exactly r times. Then $1 \leq r \leq n$. Then each such mapping takes a set of $n - r$ points *onto* a set of the form $\{2, 3, \cdots, s\}$ where $s - 1 \leq n - r \leq n - 1$. Hence, there are u_{n-r} such mappings. Since there are $\binom{n}{r}$ possible sets on which a mapping may assume the value 1 r times,

$$u_n = \sum_{r=1}^{n} \binom{n}{r} u_{n-r} = \sum_{r=0}^{n-1} \binom{n}{r} u_r \, .$$

Now $u_0 = 1 = \sum_{k=0}^{\infty} 1/2^{k+1}$. Assume, as an induction hypothesis, that $u_r = \sum_{k=0}^{\infty} k^r/2^{k+1}$. Then

$$\begin{aligned}
u_n &= \sum_{r=0}^{n-1} \binom{n}{r} u_r = \sum_{r=0}^{n-1} \binom{n}{r} \sum_{k=0}^{\infty} \frac{k^r}{2^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{r=0}^{n-1} \binom{n}{r} k^r = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} [(1+k)^n - k^n] \\
&= \sum_{k=0}^{\infty} \frac{(1+k)^n}{2^{k+1}} - \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \\
&= \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}
\end{aligned}$$

and the result follows. (The interchange of the order of summation and rearrangement of terms in the infinite sum can be justified by the absolute convergence of the series.)

39. *Second solution.* For $1 \leq i$, let v_i be the number of mappings of $\{1, 2, \cdots, n\}$ *onto* a set of exactly i elements. Observe that $v_i = 0$ when $i \geq n+1$. There are k^n mappings of $\{1, 2, \cdots, n\}$ into $\{1, 2, \cdots, k\}$, of which v_k belong to A_n. The other $k^n - v_k$ mappings will leave out i numbers in the range for some $1 \leq i \leq k-1$, and the i numbers not found can be selected in $\binom{k}{i}$ ways. Thus

$$k^n = \sum_{i=1}^{k} \binom{k}{i} v_i \quad .$$

Hence

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} &= \sum_{k=0}^{\infty} \sum_{i=1}^{k} \frac{\binom{k}{i} v_i}{2^{k+1}} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \frac{\binom{k}{i} v_i}{2^{k+1}} \\ &= \sum_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{\binom{k}{i}}{2^{k+1}} \right) v_i = \sum_{i=1}^{n} \left(\sum_{k=i}^{\infty} \frac{\binom{k}{i}}{2^{k+1}} \right) v_i \ . \end{aligned}$$

We evaluate the inner sum. Fix the positive integer i. Suppose that we flip a fair coin an indefinite number of times, and consider the event that the $(i+1)$th head occurs on the $(k+1)$th toss. Then the previous i heads could have occurred in $\binom{k}{i}$ posible positions, so that the probability of the event is $\binom{k}{i} 2^{-(k+1)}$. Since the $(i+1)$th head must occur on *some* toss with probability 1, $\sum_{k=i}^{\infty} \binom{k}{i} 2^{-(k+1)} = 1$. Hence

$$\sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = \sum_{i=1}^{n} v_i = \#A_n \ .$$

40.

40. Let $h = 2xy/(x+y)$, $g = \sqrt{xy}$, $a = \frac{1}{2}(x+y)$ and $r = \sqrt{\frac{1}{2}(x^2+y^2)}$.

40. *First solution.* It is straightforward to check that $2a^2 = r^2 + g^2$ and that $g^2 = ah$. Suppose that $h + r = a + g$. Then

$$\begin{aligned} &r = a + g - h \\ &\quad \Longrightarrow 2a^2 - g^2 = r^2 = a^2 + g^2 + h^2 - 2ah - 2gh + 2ag \\ &\quad \Longrightarrow (a+h)(a-h) = a^2 - h^2 = 2(g^2 - ah) + 2g(a-h) = 0 + 2g(a-h) \\ &\quad \Longrightarrow a = h \quad \text{or} \quad a + h = 2g \ . \end{aligned}$$

In the latter case, $g = \frac{1}{2}(a+h) = \sqrt{ah}$, so both possibilities entail $a = h$. But equality of the arithmetic and harmonic means occurs if and only if $x = y$.

40. *Second solution.* [A. Corduneanu] Observe that

$$r^2 - g^2 = \frac{1}{2}(x-y)^2 = (x+y)(a-h) = 2a(a-h) \ .$$

Since

$$\frac{r^2 - g^2}{r + g} = r - g = a - h \ ,$$

it follows that

$$a = \frac{1}{2}(r + g) \ .$$

However, it can be checked that, in general,

$$a = \sqrt{\frac{r^2 + g^2}{2}}$$

so that a is at once the arithmetic mean and the root-mean-square of r and g. But this can occur if and only if $r = g = a$ if and only if $x = y$.

40. *Third solution.* [R. O'Donnell] Because of the homogeneity of the given equation, we can assume without loss of generality that $g = 1$ so that $y = 1/x$. Then $h = 1/a$ and $r = \sqrt{2a^2 - 1}$. The equation becomes

$$r + \frac{1}{a} = a + 1$$

or

$$\sqrt{2a^2 - 1} = 1 + a - \frac{1}{a} \ .$$

Squaring and manipulating leads to

$$0 = a^4 - 2a^3 + 2a - 1 = (a-1)^3(a+1)$$

whence $a = 1 = g$ and so $x = y = 1$. The main result follows from this.

41.

41. *First solution.* For positive integer n, let x_n be the nth term of the sequence, and let $x_0 = 0$. Then, for $n \geq 0$, $x_{n+1} = 10x_n + (n+1)$ so that $x_{n+1} \equiv 3x_n + (n+1) \pmod 7$. Suppose that m is a nonnegative integer and that $x_{7m} = a$. Then

$$\begin{array}{llll} x_{7m+1} \equiv 3a+1 & x_{7m+2} \equiv 2a+5 & x_{7m+3} \equiv 6a+4 & x_{7m+4} \equiv 4a+2 \\ x_{7m+5} \equiv 5a+4 & x_{7m+6} \equiv a+4 & x_{7m+7} \equiv 3a+5 & \end{array}$$

In particular, we find that, modulo 7, $\{x_{7m}\}$ is periodic with the values $\{0, 5, 6, 2, 4, 3\}$ repeated, so that $0 \equiv x_0 \equiv x_{42} \equiv x_{84} \equiv \cdots$. Hence, modulo 7, $x_{7m+1} \equiv 0$ iff $a \equiv 2$, $x_{7m+2} \equiv 0$ iff $a \equiv 1$, $x_{7m+3} \equiv 0$ iff $a \equiv 4$, $x_{7m+4} \equiv 0$ iff $a \equiv 3$, $x_{7m+5} \equiv 0$ iff $a \equiv 2$ and $x_{7m+6} \equiv 0$ iff $a \equiv 3$. Putting this all together, we find that $x_n \equiv 0 \pmod 7$ if and only if $n \equiv 0, 22, 26, 31, 39, 41 \pmod{42}$.

42.

42. *First solution.* Produce BA to E so that $BA = AE$ and join EC. Then D is the centroid of ΔBEC and BD produced meets EC at its midpoint F. Since $AE = AC$, ΔCAE is isosceles and so $AF \perp EC$. Also, since A and F are midpoints of their respective segments, $AF \| BC$ and so $\angle AFB = \angle DBC$. Because $\angle AFC$ and $\angle APC$ are both right, $APCF$ is concyclic so that $\angle AFP = \angle ACP$.

Hence $\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle AFB = \angle ACB - \angle ACP = \angle PCB$.

42. *Second solution.* Assign coordinates: $A \sim (0, a)$, $B \sim (-1, 0)$, $C \sim (1, 0)$. Then $D \sim (\frac{1}{3}, \frac{2a}{3})$. Let $P \sim (p, q)$. Then, since P lies on the lines $y = \frac{a}{2}(x+1)$, $q = \frac{a}{2}(p+1)$. The relation $AP \perp PC$ implies that

$$-1 = \left(\frac{q-a}{p}\right)\left(\frac{q}{p-1}\right) = \left[\frac{a(p-1)}{2p}\right]\left[\frac{a(p+1)}{2(p-1)}\right] = \frac{a^2(p+1)}{4p} = \frac{aq}{2p}$$

whence $p = -a^2/(a^2+4)$ and $q = 2a/(a^2+4)$. Now

$$\tan \angle ABP = \frac{a - (a/2)}{1 + (a^2/2)} = \frac{a}{2 + a^2}$$

while

$$\tan \angle PCB = \frac{-q}{p-1} = \frac{-2a}{-a^2 - (a^2+4)} = \frac{a}{a^2+2} = \tan \angle ABP \ .$$

The result follows.