Mathematical Olympiads' Correspondence Program - 1997-1998

Problem Set 6

31. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x + f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

32. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \quad .$$

33. Let $n \geq 3$ be a positive integer. Are there n positive integers $a_1, a_2, \cdots, a_n$ not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] \quad .$$

where $S_i = a_1 + a_2 + \cdots + a_n$, and where $(\cdot, \cdot)$ and $[\cdot, \cdot]$ represent the greatest common divisor and least common multiple respectively?

34. Find all integers p for which there are rational numbers a and b such that the polynomial $x^5 - px - 1$ has at least one root in common with the polynomial $x^2 - ax - b$.

35. Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that

$$|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB| \quad .$$

(*Note:* This is Problem 2255 in *CM+MM*, September, 1997.)

36. Let n be a positive integer and $x > 0$. Prove that

$$(1 + x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n} x \quad .$$

(*Note:* This is Problem 2260 in *CM+MM*, September, 1997.)

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

31.

31. *First solution.* Setting $(x, y) = (t, 0)$ yields $f(t + f(0)) = f(t)$ for all real t. Setting $(x, y) = (0, t)$ yields $f(f(t)) = f(0) + t$ for all real t. Hence $f(f(f(t))) = f(t)$ for all real t, *i.e.*, $f(f(z)) = z$ for each z in the image of f. Let $(x, y) = (f(t), -f(0))$. Then

$$f(f(t) + f(-f(0))) = f(f(t)) - f(0) = f(0) + t - f(0) = t$$

so that the image of f contains every real and so $f(f(t)) \equiv t$ for all real t.

Taking $(x, y) = (u, f(v))$ yields

$$f(u + v) = f(u) + f(v)$$

since $v = f(f(v))$ for all real u and v. In particular, $f(0) = 2f(0)$, so $f(0) = 0$ and $0 = f(-t + t) = f(-t) + f(t)$. By induction, it can be shown that for each integer n and each real t, $f(nt) = nf(t)$. In particular, for each rational r/s, $f(r/s) = rf(1/s) = (r/s)f(1)$. Since f is continuous, $f(t) =$

$f(t \cdot 1) = tf(1)$ for all real t. Let $c = f(1)$. Then $1 = f(f(1)) = f(c) = cf(1) = c^2$ so that $c = \pm 1$. Hence $f(t) \equiv t$ or $f(t) \equiv -t$. Checking reveals that both these solutions work. (For $f(t) \equiv -t$, $f(x + f(y)) = -x - f(y) = f(x) + y$, as required.)

31. *Second solution.* Taking $(x, y) = (0, 0)$, we find that

$$f(0) = 0 + f(0) \Rightarrow f(f(0)) = f(0 + f(0)) = f(0) + f(f(0)) \Rightarrow f(0) = 0 \ .$$

Taking $(x, y) = (0, t)$ yields $f(f(t)) = t$ for all real t. Hence $f(x + y) = f(x + f(f(y)) = f(x) + f(y)$ for all x, y, and we can complete the solution as in the First Solution.

31. *Third solution.* Taking $(x, y) = (0, 0)$ yields $f(f(0)) = f(0)$, whence $f(f(f(0))) = f(f(0)) = f(0)$. Taking $(x, y) = (0, f(0))$ yields $f(f(f(0))) = 2f(0)$. Hence $2f(0) = f(0)$ so that $f(0) = 0$. Taking $x = 0$ yields $f(f(y)) = y$ for each y. We can complete the solution as in the Second Solution.

32.

32. *First solution.* Let $N = n^{n^{n^n}} - n^{n^n}$. Since $1989 = 3^2 \cdot 13 \cdot 17$,

$$N \equiv 0 \pmod{1989} \Leftrightarrow N \equiv 0 \quad (\text{mod} \quad 9, 13 \ \& \ 17) \ .$$

We require the following facts:
(i) $x^u \equiv 0$ (mod 9) whenever $u \geq 2$ and $x \equiv 0$ (mod 3).
(ii) $x^6 \equiv 1$ (mod 9) whenever $x \not\equiv 0$ (mod 3).
(iii) $x^u \equiv 0$ (mod 13) whenever $x \equiv 0$ (mod 13).
(iv) $x^{12} \equiv 1$ (mod 13) whenever $x \not\equiv 0$ (mod 13), by Fermat's Little Theorem.
(v) $x^u \equiv 0$ (mod 17) whenever $x \equiv 0$ (mod 17).
(vi) $x^{16} \equiv 1$ (mod 17) whenever $x \not\equiv 0$ (mod 17), by FLT.
(vii) $x^4 \equiv 1$ (mod 16) whenever $x = 2y + 1$ is odd. (For, $(2y + 1)^4 = 16y^3(y + 2) + 8y(3y + 1) + 1 \equiv 1$ (mod 16).)

Note that

$$N = n^{n^n} \left[n^{(n^{n^n} - n^n)} - 1 \right] = n^{n^n} \left[n^{n^n (n^{n^n - n} - 1)} - 1 \right] \ .$$

Modulo 17. If $n \equiv 0$ (mod 17), then $n^{n^n} \equiv 0$, and so $N \equiv 0$ (mod 17).
If n is even, $n \geq 4$, then $n^n \equiv 0$ (mod 16), so that

$$n^{n^n (n^{n^n - n} - 1)} \equiv 1^{(n^{n^n - n} - 1)} \equiv 1$$

so $N \equiv 0$ (mod 17).
Suppose that n is odd. Then $n^n \equiv n$ (mod 4)

$$\Rightarrow n^n - n = 4r \quad \text{for} \quad \text{some} \quad r \in \mathbf{N}$$

$$\Rightarrow n^{n^n - n} = n^{4r} \equiv 1 \pmod{16}$$

$$\Rightarrow n^{n^n - n} - 1 \equiv 0 \pmod{16}$$

$$\Rightarrow n^{n^n (n^{n^n - n} - 1)} \equiv 1 \quad (\text{mod } 17$$

$$\Rightarrow N \equiv 0 \pmod{17} \ .$$

Hence $N \equiv 0$ (mod 17) for all $n \geq 3$.

Modulo 13. If $n \equiv 0$ (mod 13), then $n^{n^n} \equiv 0$ and $N \equiv 0$ (mod 13).

Suppose that n is even. Then $n^n \equiv 0 \pmod 4$, so that $n^{n^n} - n^n \equiv 0 \pmod 4$. Suppose that n is odd. Then $n^{n^n - n} - 1 \equiv 0 \pmod{16}$ and so $n^{n^n} - n^n \equiv 0 \pmod 4$.
If $n \equiv 0 \pmod 3$, then $n^n \equiv 0$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. If $n \equiv 2 \pmod 3$, then, as $n^n - n$ is always even, $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. Hence, for all n, $n^{n^n} - n^n \equiv 0 \pmod 3$.
It follows that $n^{n^n} - n^n \equiv 0 \pmod{12}$ for all values of n. Hence, when n is not a multiple of 13, $n^{(n^{n^n}-n)} \equiv 1$ so $N \equiv 0 \pmod{13}$.

Modulo 9. If $n \equiv 0 \pmod 3$, then $n^{n^n} \equiv 0 \pmod 9$, so $N \equiv 0 \pmod 9$. Let $n \not\equiv 0 \pmod 9$. Since $n^{n^n} - n^n$ is divisible by 12, it is divisible by 6, and so $n^{(n^{n^n}-n^n)} \equiv 1$ and $N \equiv 0 \pmod 9$. Hence $N \equiv 0 \pmod 9$ for all n.

The required result follows.

33.

33. *First solution.* Letting $b_i = (a_i, S_i)$, we find that

$$[a_i, S_i] = \frac{a_i S_i}{(a_i, S_i)} = \frac{a_i S_i}{b_i} \ .$$

The given condition is equivalent to $a_i + S_i = b_i + (a_i S_i / b_i)$, which is equivalent to

$$0 = b_i^2 - (a_i + S_i) b_i + a_i S_i = (b_i - a_i)(b_i - S_i) \ .$$

We can achieve the condition by making $a_i = (a_i, S_i)$ and $S_i = [a_i, S_i]$. Let $a_1 = a_2 = 1$, $a_i = 2^{i-2}$ for $i \geq 3$. Then

$$S_i = 1 + \sum_{j=2}^{i} 2^{j-2} = 1 + (2^{i-1} - 1) = 2^{i-1}$$

$$\Longrightarrow (a_i, S_i) = 2^{i-2} \ , \quad [a_i, S_i] = 2^{i-1}$$

for $i \geq 3$.

33. *Second solution.* Let $a_i = 1$, $a_2 = 2$ and $a_i = 3 \cdot 2^{i-3}$ for $i \geq 3$. Then

$$S_i = 1 + 2 + 3 \sum_{j=0}^{i-3} 2^j = 1 + 2 + 3(2^{i-2} - 1) = 3 \cdot 2^{i-2}$$

$$\Longrightarrow (a_i, S_i) = 3 \cdot 2^{i-3} \ , \quad [a_i, S_i] = 3 \cdot 2^{i-2}$$

for $i \geq 3$.

33. *Third solution.* [K. Purbhoo] Choose a_1 at will, and let $a_i = S_{i-1}$ for $i \geq 2$. Then $S_i = S_{i-1} + a_i = 2a_i$, $a_i + S_i = 3a_i$, $(a_i, S_i) = a_i$ and $[a_i, S_i] = 2a_i$ for $i \geq 2$.

33. *Fourth solution.* [K. Yeats] Let $a_1 = 1$, $a_2 = 3$ and $a_n = 2^{n-1}$ for $n \geq 3$. Then $S_n = 2^n$ for $n \geq 2$ and, for each i with $3 \leq i \leq n$, $(a_i, S_i) = 2^{i-1}$ and $[a_i, S_i] = 2^i$.

34.

34. *First solution.* A common rational root must be 1 or -1, in which case the respective values of p are 0 and 2. These roots are shared with $x^2 - 1$ ($a = 0$, $b = 1$). Suppose that there is a common nonrational root r. Then $r^2 = ar + b$ so that

$$r^4 = a^2 r^2 + 2abr + b^2 = (a^3 + 2ab) r + (a^2 b + b^2)$$

whence

$$pr + 1 = r^5 = (a^3 + 2ab)r^2 + (a^2b + b^2)r = (a^4 + 3a^2b + b^2)r + b(a^3 + 2ab) \ .$$

Since r is not rational,

$$p = a^4 + 3a^2b + b^2 \quad \text{and} \quad 1 = b(a^3 + 2ab) \ .$$

Hence $2ap - 1 = 2a^5 + 5a^3b$, so that $5a^3b = 2ap - 2a^5 - 1$. Thus

$$1 = \left(\frac{-2a^5 + 2ap - 1}{5a^3}\right)\left(a^3 + \frac{-4a^5 + 4pa - 2}{5a^2}\right)$$

from which we obtain that

$$25a^5 = (-2a^5 + 2pa - 1)(a^5 + 4pa - 2) \ .$$

This simplifies to

$$a^{10} + 3pa^6 + 11a^5 - 4p^2a^2 + 4pa - 1 = 0 \ .$$

This is a polynomial equation for a with integer coefficients. In order that a be rational, it must be an integer dividing 1. Hence $a = \pm 1$. If $a = 1$, then $1 = b(1 + 2b)$, whence $b = \frac{1}{2}$ or $b = -1$. The first value does not give an integer value of p, but the second yields $p = -1$. In this case, the two polynomials are $x^2 - x + 1$ and $x^5 + x - 1 = (x^2 - x + 1)(x^3 + x^2 - 1)$ with common roots $-\omega$ and $-\omega^2$, where ω is an imaginary cube root of unity. If $a = -1$, we get the equation $2b^2 + b + 1 = 0$ which is satisfied by no rational value of b. Thus the possibilities are $p = -1, 0, 2$.

34. *Second solution.* The rational root case can be handled as in the first solution. By long division, we find that

$$x^5 - px - 1 = (x^2 - ax - b)[x^3 + ax^2 + (a^2 + b)x + (a^3 + 2ab)]$$

$$+(a^4 + 3a^2b + b^2 - p)x + (a^3b + 2ab^2 - 1) \ .$$

If r is a nonrational root of the quadratic and the quintic, then its quadratic conjugate is also. So the quadratic must divide the quintic and both coefficients of the foregoing division must vanish. Hence $p = a^4 + 3a^2b + b^2$ and $1 = b(a^3 + 2ab)$, and we can proceed as in the first solution.

35.

35. *First solution.* The result is clear if P is on the bisector of the angle at A, since both sides of the inequality are 0.

Wolog, let P be closer to AB than AC, and let Q be the image of P under reflection in the bisector of the angle A. Then

$$\angle PAQ = \angle PAC - \angle QAC = \angle PAC - \angle PAB$$

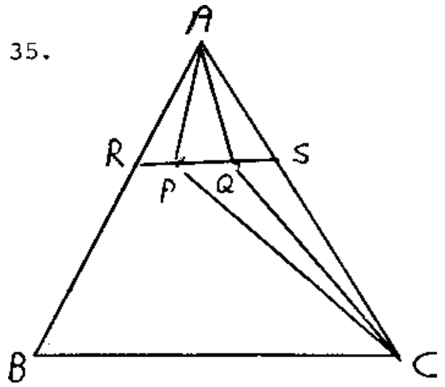
and

$$\angle PCQ = \angle QCB - \angle PCB = \angle PBC - \angle PCB \ .$$

Thus, it is required to show that $\angle PAQ \geq \angle PCQ$.

Produce PQ to meet AB in R and AC in S. Consider the reflection $\mathcal{R}$ with axis RS. The circumcircle $\mathcal{C}$ of ΔARS is carried to a circle $\mathcal{C}'$ with chord RS. Since $\angle RCS < 60^\circ = \angle RAS$ and the angle subtended at the major arc of $\mathcal{C}'$ by RS is 60°, the point C must lie outside of $\mathcal{C}'$. The circumcircle $\mathcal{D}$ of ΔAPQ is carried by $\mathcal{R}$ to a circle $\mathcal{D}'$ with chord PQ. Since $\mathcal{D}$ is contained in $\mathcal{C}$, $\mathcal{D}'$ must be contained in $\mathcal{C}'$, so C must lie outside of $\mathcal{D}'$. Hence $\angle PQC$ must be less than the angle subtended at the major arc of $\mathcal{D}'$ by PQ, and this angle is equal to $\angle PAQ$. The result follows.

35.



36.

36. *First solution.* By the Arithmetic-Geometric Means Inequality, we have that

$$\frac{1+x}{n+1} = \frac{n(1/n)+x}{n+1} \geq \left[\left(\frac{1}{n}\right)^n x\right]^{\frac{1}{n+1}}$$

so that

$$\frac{(1+x)^{n+1}}{(n+1)^{n+1}} \geq \frac{x}{n^n}$$

and the result follows.

36. *Second solution* (by calculus). Let

$$f(x) = n^n(1+x)^{n+1} - (n+1)^{n+1}x \quad \text{for} \quad x > 0\ .$$

Then

$$f'(x) = (n+1)[n^n(1+x)^n - (n+1)^n] = (n+1)n^n[(1+x)^n - (1+\frac{1}{n})^n]$$

so that $f'(x) < 0$ for $0 < x < 1/n$ and $f'(x) > 0$ for $1/n < x$. Thus $f(x)$ attains its minimum value 0 when $x = 1/n$ and so $f(x) \geq 0$ when $x > 0$. The result follows.

36. *Third solution.* Let $u = nx - 1$ so that $x = (1+u)/n$. Then

$$\begin{aligned}(1+x)^{n+1} - \frac{(n+1)^{n+1}}{n^n}x &= (1+\frac{1}{n}+\frac{u}{n})^{n+1} - (1+\frac{1}{n})^{n+1}(1+u) \\ &= (1+\frac{1}{n})^{n+1} + (n+1)(1+\frac{1}{n})^n\frac{u}{n} + \binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 \\ &\quad + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots - (1+\frac{1}{n})^{n+1}(1+u) \\ &= \binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots\ .\end{aligned}$$

This is clearly nonnegative when $u \geq 0$. Suppose that $-1 < u < 0$. For $1 \leq k \leq n/2$, we have that

$$\binom{n+1}{2k}(1+\frac{1}{n})^{n-2k+1}(\frac{u}{n})^{2k} + \binom{n+1}{2k+1}(1+\frac{1}{n})^{n-2k}(\frac{u}{n})^{2k+1}$$

$$= \frac{(n+1)!(1+1/n)^{n-2k}}{(2k+1)!(n+1-2k)!}\left(\frac{u}{n}\right)^{2k}[(2k+1)(1+\frac{1}{n}) + (n+1-2k)(\frac{u}{n})]\ .$$

This will be nonnegative if and only if the quantity in square brackets is nonnegative. Since $u > -1$, this quantity exceeds

$$(2k+1)(1+\frac{1}{n}) - (n+1-2k)(\frac{1}{n}) = \left(\frac{n+1}{n}\right)(2k+1-1) - \frac{2k}{n} = 2k > 0 \ .$$

Thus, each consecutive pair of terms in the sequence

$$\binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots$$

has a positive sum and so the desired result follows.