Mathematical Olympiads' Correspondence Program - 1997-1998

Problem Set 4

19. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n-r) : 1 \leq r \leq n-1\}$. Determine necessary and sufficient conditions on the pair (a, b) that $f(a+b) = f(a) + f(b)$.

20. Suppose that a and c are fixed real numbers with $a \leq 1 \leq c$. Determine the largest value of b which is compatible with $a + bc \leq b + ac \leq c + ab$.

21. Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$

$$\sqrt{x} + \sqrt{y} = 2 \ .$$

22. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$ and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3CD$.

23. Two unequal spheres in contact have a common tangent cone. The three surfaces divide space into various parts, only one of which is bounded by all three surfaces; it is "ring-shaped". Being given the radii r and R of the spheres with $r < R$, find the volume of the "ring-shaped" region in terms of r and R.

24. Let ABC be a triangle. Select points D, E, F outside of ΔABC such that ΔDBC, ΔEAC, ΔFAB are all isosceles with the equal sides meeting at these outside points and with $\angle D = \angle E = \angle F$. Prove that the lines AD, BE and CF all intersect in a common point.

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

19.

19. *First solution.* From the first few values of $f(n)$, we conjecture that $f(2k) = 3k$ and $f(2k+1) = 3k+1$ for each positive integer k. We establish this by induction. It is easily checked for $k = 1$. Suppose that it holds up to $k = m$.

Suppose that $2m+2$ is the sum of two positive even numbers $2x$ and $2y$. Then $f(2x)+f(2y) = 3(x+y) = 3(m+1)$. If $2m+2$ is the sum of two positive odd numbers $2u+1$ and $2v+1$, then

$$f(2u+1) + f(2v+1) = (3u+1) + (3v+1) = 3(u+v) + 2 < 3(u+v+1) = 3(m+1) \ .$$

Hence $f(2(m+1)) = 3(m+1)$.

Suppose $2m+3$ is the sum of $2z$ and $2w+1$. Then $z + w = m+1$ and

$$f(2z) + f(2w+1) = 3z + 3w + 1 = 3(z+w) + 1 = 3(m+1) + 1 \ .$$

Hence $f(2(m+1)+1) = 3(m+1)+1$. The conjecture is established by induction.

By checking cases on the parity of a and b, one verifies that $f(a+b) = f(a)+f(b)$ if and only if at least one of a and b is even. (If a and b are both odd, the left side is divisible by 3 while the right side is not.)

19. *Second solution.* [K. Yeats] By inspection, we conjecture that $f(n+1) = f(n)+2$ when n is odd, and $f(n+1) = f(n)+1$ when n is even. This is true for $n = 1, 2$. Suppose it holds up to $n = 2k$. If $2k+1 = i+j$ with i even and j odd, then $f(i-1)+f(j+1) = f(i)-2+f(j)+2 = f(i)+f(j)$ and $f(i+1)+f(j-1) = f(i)+1+f(j)-1 = f(i)+f(j)$ (where defined), so in particular $f(2k+1) =$

$f(2k)+f(1) = f(2k)+1$. Note that this also tells us that $f(2k+1) = f(i)+f(j)$ whenever $i+j = 2k+1$. Now consider $2k+2 = i+j$. If i and j are both even, then

$$f(i+1) + f(j-1) = f(i) + 1 - f(j) - 2 = f(i) + f(j) - 1$$

while if i and j are both odd, then

$$f(i+1) + f(j-1) = f(i) + 2 - f(j) - 1 = f(i) + f(j) + 1 \; .$$

Thus, $f(2k+2) = f(i)+f(j)$ if and only if i and j are both even. In particular, $f(2k+2) = f(2k)+f(2) = f(2k+1) - 1 + 3 = f(2k) + 2$. We thus find that $f(a+b) = f(a) + f(b)$ if and only if at least one of a and b is even.

20.

20. *First solution.* Observe that $(b+ac) - (a+bc) = (c-1)(a-b)$ and $(c+ab) - (b+ac) = (1-a)(c-b)$, so that the inequalities are equivalent to

$$(c-1)(a-b) \geq 0 \qquad \text{and} \qquad (1-a)(c-b) \geq 0 \; .$$

If $a = c = 1$, then the inequalities hold for all values of b and so there is no largest value of b. If $a = 1 < c$, then the inequalities hold if and only if $b \leq a = 1$, so that largest value of b is 1. If $a < 1 = c$, then the inequalities hold if and only if $b \leq c = 1$, so the largest value of b is 1. Finally, if $a < 1 < c$, then the inequalities hold if and only if $b \leq a$, so the largest value of b is a.

21.

21. *Preliminary comments.* With the surds in the second equation, we must restrict ourselves to nonnegative values of x. Because of the complexity of the expressions, it is probably impossible to eliminate one of the variables and solve for the other. Let us make a few preliminary observations:

(i) $(x, y) = (1, 1)$ is an obvious solution;

(ii) Both equations are symmetric in x and y;

(iii) Taking $f(x, y) = 2^{x^2+y} + 2^{x+y^2}$ and $g(x, y) = \sqrt{x} + \sqrt{y}$, we have that $f(0, y) = 2^y + 2^{y^2}$ and $g(0, y) = \sqrt{y}$; thus, $f(0, y) = 8 \Rightarrow 1 < y < 2$ and $g(0, y) = 2 \Leftrightarrow y = 4$. The graphs of $f(x, y) = 8$ and $g(x, y) = 2$ may look like the figures in the diagram.

This suggests that $f(x, y) = 8 \Rightarrow x + y \leq 2$ and $g(x, y) = 2 \Rightarrow x + y \geq 2$ with equality for both $\Leftrightarrow (x, y) = (1, 1)$. Hence we look for a relationship among $f(x, y)$, $g(x, y)$ and $x + y$.

First solution.

$$(\sqrt{x} + \sqrt{y})^2 = x + 2\sqrt{xy} + y \leq x + (x+y) + y = 2(x+y)$$

by the Arithmetic-Geometric Means Inequality. Hence

$$\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)} \; .$$

Also, by the same AGM inequality,

$$2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{x^2+y+x+y^2}} \quad .$$

Now, using the inequality again, we find that

$$x^2 + y + x + y^2 = (x^2 + y^2) + (x+y) \geq \frac{1}{2}(x+y)^2 + (x+y)$$

so that

$$2^{x^2+y} + 2^{x+y^2} \geq 2^{1+\frac{1}{4}(x+y)^2+\frac{1}{2}(x+y)} = 2^{\frac{1}{4}[(x+y+1)^2+3]} .$$

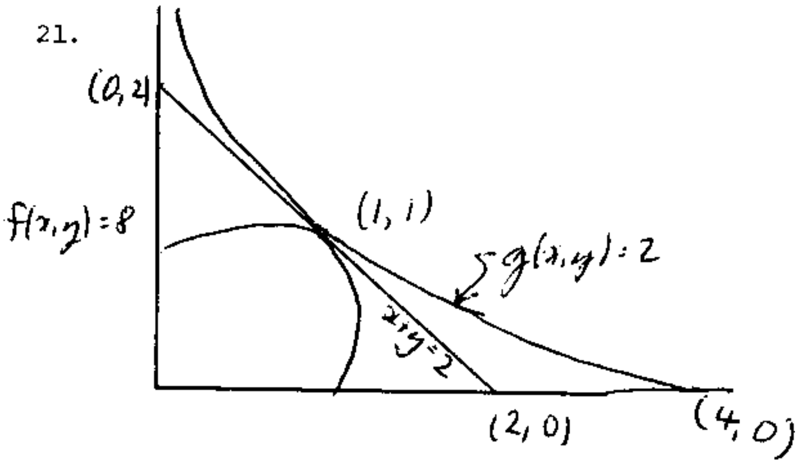
Suppose the (x, y) satisfies the system. Then

$$\sqrt{2(x+y)} \geq 2 \Rightarrow (x+y) \geq 2$$

and

$$\frac{1}{4}[(x+y+1)^2+3] \leq 3 \Rightarrow (x+y+1)^2 \leq 9 \Rightarrow x+y+1 \leq 3 \Rightarrow x+y \leq 2 \ .$$

Hence $x + y = 2$ and all inequalities are equalities. Therefore $x = y = 1$.



21. *Second solution.* [A. Rodriguez] Wolog, we may assume that $x \geq 1$. Let $\sqrt{x}+\sqrt{y} = 2$; then $y = (2-\sqrt{x})^2$. Define

$$\begin{aligned} g(x) &= x + y^2 + y + x^2 = (2-\sqrt{x})^4 + x^2 + x + (2-\sqrt{x})^2 \\ &= 2x^2 - 8x^{\frac{3}{2}} + 26x - 36x^{\frac{1}{2}} + 20 \ . \end{aligned}$$

Then

$$\begin{aligned} g'(x) &= 4x - 12x^{\frac{1}{2}} + 26 - 18x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}(2x^{\frac{3}{2}} - 6x + 13x^{\frac{1}{2}} - 9) \\ &= 2x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1)(2x - 4x^{\frac{1}{2}} + 9) = 2x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1)[2(x^{\frac{1}{2}} - 1)^2 + 7] > 0 \end{aligned}$$

for $x > 1$. Hence $g(x)$ is strictly increasing for $x > 1$, so that $g(x) \geq g(1) = 4$ for $x \geq 1$ with equality if and only if $x = 1$. Thus, if the first equation holds, then

$$8 = 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{g(x)}} \Rightarrow 16 \geq 2^{g(x)} \Rightarrow g(x) \leq 4 \ .$$

Hence $g(x) = 4$, so that $x = 1$ and $y = 1$. Thus, $(x, y) = (1, 1)$ is the only solution.

21. *Third solution.* [S. Yazdani] Set $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$. Then $x^2 + y = (1+u)^4 + (1-u)^2$ and $x + y^2 = (1-u)^4 + (1+u)^2$, so

$$8 = 2^{x^2+y} + 2^{x+y^2} = 2^{u^4+7u^2+2}\left(2^{4u^3+2u} + \frac{1}{2^{4u^3+2u}}\right) \geq 2^2(2) = 8$$

with equality if and only if $u = 0$. Since the extremes of this inequality are equal, we must have $u = 0$, so $x = y = 1$.

21. *Fourth solution.* [C. Hsia] With $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$, we can write the first equation as

$$2^{4u^3+2u} + \frac{1}{2^{4u^3+2u}} = 2^{1-7u^2-u^4} \ .$$

Let $z = 2^{4u^3+2u}$. We note that the quadratic $z^2 - 2^{1-7u^2-u^4}z + 1 = 0$ is solvable, and so has nonnegative discriminant. Hence

$$2^{2-14u^2-2u^4} \geq 4 = 2^2 \Rightarrow -14u^2 - 2u^4 \geq 0 \Rightarrow u = 0 \quad .$$

Hence $x = y = 1$.

21. *Fifth solution.* [M. Boase] $2(x+y) \geq (x+y) + 2\sqrt{xy} = (\sqrt{x} + \sqrt{y})^2 = 4$ so that $x + y \geq 2$. Let $f(t) = t(t+1)$. For positive values of t, $f(t)$ is an increasing strictly convex function of t. Hence

$$f(x) + f(y) \geq 2f(\frac{1}{2}(x+y)) \geq 2f(1) = 4$$

so that $x^2+x+y^2+y \geq 4$. Equality occurs if and only if $x = y = 1$. Applying the Arithmetic-Geometric Means Inequality, we find that

$$4 = \frac{1}{2}(2^{x^2+y} + 2^{x+y^2}) \geq 2^{\frac{1}{2}(x^2+y^2+x+y)}$$

so that $x^2 + x + y^2 + y \leq 4$. Hence $x^2 + x + y^2 + y = 4$ and so $x = y = 1$.

Comment. Note that $2(x^2 + y^2) \leq (x+y)^2$ with equality if and only if $x = y$. Hence

$$x^2 + y^2 + x + y \geq \frac{1}{2}(x+y)^2 + (x+y) \geq 4$$

with equality if and only if $x = y = 1$. This avoids the use of the convexity of the function f.

21. *Sixth solution.* [J. Chui] Wolog, let $x \geq y$ so that $\sqrt{x} \geq 1 \geq \sqrt{y}$. Suppose that $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$. Then $x + y = 2 + 2u^2 \geq 2$ and $xy = (1-u^2)^2 \leq 1$. Thus

$$\begin{aligned} 8 &= 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{x^2+y+x+y^2}} \\ &= 2\sqrt{2^{(x+y)(x+y+1)-2xy}} \geq 2\sqrt{2^{2\cdot 3 - 2\cdot 1}} = 2^3 = 8 \end{aligned}$$

with equality if and only if $x = y$.

22.

22. *First solution.* Case (i): Suppose that $\angle B$ is acute. Let $AH \perp BC$ and E lie on CH such that $AE = AB$. $AC^2 - CH^2 = AB^2 - BH^2$ implies that

$$AC^2 - AB^2 = CH^2 - BH^2 = (CH-BH)(CH+BH) = (CH-HE)BC = CE \cdot BC = CE[2(AC-AB)] \; .$$

Hence $AC + AB = 2CE$. Also $AC - AB = \frac{1}{2}BC$. Therefore $2AB + \frac{1}{2}BC = 2CE$.

Suppose that $\angle ABD = 2\angle ADB$. Then $\angle AEB = 2\angle ADB \Rightarrow \Delta ADE$ is isosceles. Hence

$$AB = AE = DE \Rightarrow 2DE + \frac{1}{2}BC = 2CE \Rightarrow BC = 4(CE - DE) = 4CD \Rightarrow BD = 3CD \; .$$

Conversely, suppose that $BD = 3CD$. Then

$$BC = 4CD \Rightarrow \frac{1}{4}BC = CE - DE \; .$$

From the above,

$$\begin{aligned} AB = CE - \frac{1}{4}BC = DE \Rightarrow AE = DE \\ \Rightarrow \angle ABD = \angle AEB = 2\angle ADB \; . \end{aligned}$$

Case (ii): Suppose $\angle B = 90°$. Then

$$AC^2 - AB^2 = BC^2 = 2(AC - AB) \cdot BC \Rightarrow AC + AB = 2BC$$

$$\Rightarrow \frac{1}{2}BC + AB + AB = 2BC \Rightarrow AB = \frac{3}{4}BC \quad .$$

$$\angle ABD = 2\angle ADB \Rightarrow \angle ADB = 45° = \angle BAD \Rightarrow AB = BD$$

$$\Rightarrow BD = \frac{3}{4}BC \Rightarrow BD = 3CD \ .$$

$$BD = 3CD \Rightarrow BD = \frac{3}{4}BC = AB \Rightarrow \angle ADB = \angle BAD = 45° = \frac{1}{2}\angle ABD \quad .$$

Case (iii): Suppose $\angle B$ exceeds $90°$. Let $AH \perp BC$ and E be on CH produced such that $AE = AB$. Then

$$AC^2 - CH^2 = AB^2 - BH^2 \Rightarrow (AC - AB)(AC + AB) = CH^2 - BH^2 = (CH - BH)(CH + BH) = CB \cdot CE$$

$$\Rightarrow AC + AB = 2CE \ .$$

Also

$$AC - AB = \frac{1}{2}BC \Rightarrow 2AB + \frac{1}{2}BC = 2CE \Rightarrow AB + \frac{1}{4}BC = CE \quad .$$

Let $\angle ABD = 2\angle ADB$. Then

$$180° - \angle ABE = 2\angle ADB \Rightarrow \angle AEB + 2\angle ADE = \angle ABE + 2\angle ADB = 180° \ .$$

Also

$$\angle AEB + \angle EAD + \angle ADE = 180° \Rightarrow \angle EAD = \angle ADE \Rightarrow AE = ED \quad .$$

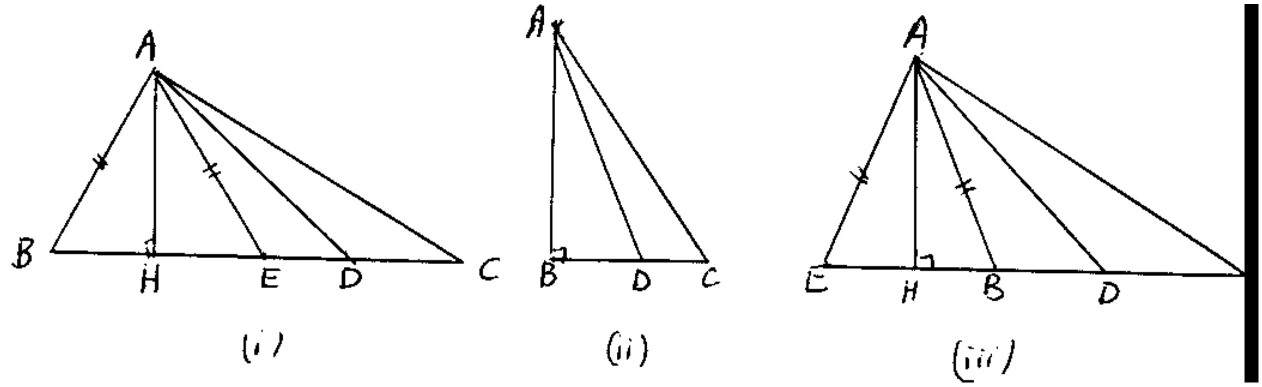
Hence

$$AB = ED \Rightarrow 2ED + \frac{1}{2}BC = 2CE \Rightarrow BC = 4(CE - DE) = 4CD \Rightarrow BD = 3CD \quad .$$

Conversely, suppose that $BD = 3CD$. Then $BC = 4CD$ and $ED = CE - CD = CE - \frac{1}{4}BC = AB$ so that $ED = AE$ and $\angle EAD = \angle ADE$. Therefore

$$\angle ABD = 180° - \angle AED = \angle EAD + \angle ADE = 2\angle ADE = 2\angle ADB \quad .$$

22.



22. *Second solution.* [R. Hoshino] Let $\angle ABD = 2\theta$. By the Law of Cosines, with the usual conventions for a, b, c,

$$\begin{aligned}
1 - 2\sin^2\theta = \cos 2\theta &= \frac{c^2 + 4(b-c)^2 - b^2}{4c(b-c)} \\
&= \frac{b-c}{c} - \frac{b+c}{4c} = \frac{3b-5c}{4c} \quad (\text{since } \; b \neq c)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 3(b-c) = 6c - 8c\sin^2\theta \\
&\Rightarrow \frac{3(b-c)}{2}\sin\theta = c(3\sin\theta - 4\sin^3\theta) = c\sin 3\theta \\
&\Rightarrow \frac{\sin\theta}{c} = \frac{2\sin 3\theta}{3(b-c)} \quad . \; (*)
\end{aligned}$$

Suppose now that D is selected so that $\angle ADB = \theta$. Then, by the Law of Sines,

$$\frac{\sin\theta}{c} = \frac{\sin(180^\circ - 3\theta)}{x} = \frac{\sin 3\theta}{x}$$

where $x = |BD|$. Comparison with (*) yields $x = \frac{1}{2}(3(b-c))$ so $4BD = 3BC \Rightarrow BD = 3CD$ as desired. On the other hand, suppose D is selected so that $BD = 3CD$. Then $BD = \frac{3}{2}(b-c)$. Let $\angle ADB = \phi$. Then

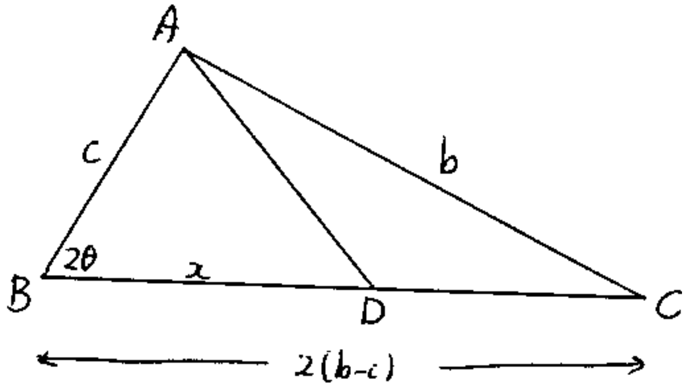
$$\frac{\sin\phi}{c} = \frac{\sin(180^\circ - \phi - 2\theta)}{\frac{3}{2}(b-c)} = \frac{\sin(\phi + 2\theta)}{\frac{3}{2}(b-c)} \; .$$

Hence

$$\begin{aligned}
\frac{\sin(\phi + 2\theta)}{\sin\phi} = \frac{\sin 3\theta}{\sin\theta} &\Rightarrow \sin\theta \sin(\phi + 2\theta) = \sin 3\theta \sin\phi \\
&\Rightarrow \frac{1}{2}[\cos(\theta + \phi) - \cos(3\theta + \phi)] = \frac{1}{2}[\cos(3\theta - \phi) - \cos(3\theta + \phi)] \\
&\Rightarrow \cos(\theta + \phi) = \cos(3\theta - \phi) \\
&\Rightarrow \theta + \phi = \pm(3\theta - \phi) \quad \text{ or } \quad \theta + \phi + 3\theta - \phi = 360^\circ \; .
\end{aligned}$$

The only viable possibility is $\theta + \phi = 3\theta - \phi \Rightarrow \theta = \phi$ as desired.

22.



22. *Third solution.* [J. Chui] First, recall *Stewart's Theorem.* Let XYZ be a triangle with sides x, y, z respectively opposite XYZ. Let W be a point on YZ so that $|XW| = u$, $|YW| = v$ and $|ZW| = w$. Then $x(u^2 + vw) = vy^2 + wz^2$. This is an immediate consequence of the Law of Cosines. Let $\theta = \angle YWX$. Then $z^2 = u^2 + v^2 - 2uv\cos\theta$ and $y^2 = u^2 + w^2 + 2uw\cos\theta$. Multiply these equations by u and v respectively, add and use $x = v + w$ to obtain the result.

Now to the problem. Suppose $BD = 3CD$. Let $|AC| = 2b$, $|AB| = 2c$, so that $|BC| = 4(b-c)$, $|BD| = 3(b-c)$ and $|CD| = b-c$. If $|AD| = d$, then an application of Stewart's Theorem yields $d^2 = 2c(3b-c)$. Applying the Law of Cosines to ΔABC and ΔABD respectively yields

$$\cos\angle ABC = \frac{3b-5c}{4c} \quad \text{and} \quad \cos\angle ADB = \frac{3b-c}{2\sqrt{2c(3b-c)}} \quad .$$

Then $\cos 2\angle ADB = (3b-5c)/4c$. Hence, either $2\angle ADB = \angle ABC$ or $\angle ABC + 2\angle ADB = 360°$. In the latter case, $\angle ABC + \angle ADB = 360° - \angle ADB > 180°$, which is false. Hence $\angle ABC = 2\angle ADB$.

On the other hand, let $2\angle ADB = \angle ABC$. If D' is a point on BC with $BD' = 3CD'$, the $2\angle AD'B = \angle ABC = 2\angle ADB$, so that $D = D'$. The result follows.

22. *Fourth solution.* Let $|AB| = a$, $|AC| = a+2$, $|BD| = 3$, $|CD| = 1$, $\angle ABD = 2\theta$, $\angle ADB = \phi$. Then $(a+2)^2 = a^2 + 16 - 8a\cos 2\theta$, whence $a = 3(1+2\cos 2\theta)^{-1}$ (so $0 < \theta < 60°$). By the Law of Sines,

$$\frac{\sin(2\theta+\phi)}{3} = \frac{(1+2\cos 2\theta)\sin\phi}{3}$$

so that

$$\begin{aligned} 0 &= \sin\phi + 2\sin\phi\cos 2\theta - \sin(2\theta+\phi) \\ &= \sin\phi + \sin\phi\cos 2\theta - \sin 2\theta\cos\phi \\ &= \sin\phi + \sin(\phi - 2\theta) = 2\sin(\phi-\theta)\cos\theta \quad . \end{aligned}$$

Since $0 \leq |\phi - \theta| < 180°$, we find that $\phi = \theta$ as desired. The converse can be obtained as in the third solution.

22. *Fifth solution.* [A. Birka] First, note that, when $BD = 3CD$, we must have $\angle ADB < 90°$, since $AC > AB$ and D is on the same side of the altitude from A as C. Also, when $\angle ABD = 2\angle ADB$, $\angle ADB < 90°$. Thus, we can assume that $\angle ADB$ is acute throughout.

We can select positive numbers u, v and w so that $|BC| = v+w$, $|AC| = u+w$ and $|AB| = u+v$. By hypothesis, $v + w = 2(w-v)$, so that $w = 3v$.

Suppose that $BD = 3CD$. Then $BC = 4CD$, whence $|CD| = v$. Hence $|BD| = 3v$. By the Law of Cosines,

$$(u+3v)^2 = (u+v)^2 + (4v)^2 - 8v(u+v)\cos B$$

so that

$$\cos B = \frac{8v^2 - 4uv}{8v(u+v)} = \frac{2v-u}{2(u+v)} \quad .$$

Hence

$$|AD|^2 = (u+v)^2 + (3v)^2 - 6v(u+v)\cos B = u^2 + 5uv + 4v^2 = (u+4v)(u+v) \quad .$$

Since $\sin^2\angle ABD = 1 - \cos^2 B = [3u(u+4v)]/[4(u+v)^2]$, and, by the Law of Sines,

$$\frac{\sin^2\angle ADB}{\sin^2\angle ABD} = \frac{u+v}{u+4v} \ ,$$

we have that

$$\sin^2\angle ADB = \frac{3u}{4(u+v)} \quad \text{and} \quad \cos^2\angle ADB = \frac{u+4v}{4(u+v)} \quad .$$

Thus $\sin^2\angle ABD = 4\sin^2\angle ADB\cos^2\angle ADB$ so that either $\angle ABD = 2\angle ADB$ or $\angle ABD + 2\angle ADB = 180°$. The latter case would yield $\angle ADB = \angle BAD$, so that $AB = BD$. This would make ΔABC a $3-4-5$ right triangle and ΔABD an isosceles right triangle, whence $90° = \angle ABD = 2\angle ADB$. The converse can be shown as in the previous solutions. The result follows.

23. *First solution.* The configuration described is obtained by rotating this configuration about the line passing through the centres P and Q of the spheres.

From a consideration of similar triangles and pythagoras theorem, we find that

$$|OP| = r\left(\tfrac{R+r}{R-r}\right) \qquad |OU| = \tfrac{4Rr^2}{R^2-r^2}$$
$$|UP| = r\left(\tfrac{R-r}{R+r}\right) \qquad |AU| = \tfrac{2r}{R+r}\sqrt{Rr}$$
$$|OQ| = R\left(\tfrac{R+r}{R-r}\right) \qquad |OV| = \tfrac{4R^2r}{R^2-r^2}$$
$$|VQ| = R\left(\tfrac{R-r}{R+r}\right) \qquad |BV| = \tfrac{2R}{R+r}\sqrt{Rr}$$

The volume of the cone obtained by rotating OBV is

$$\frac{1}{3}\pi|BV|^2|OV| = \frac{16\pi R^5 r^2}{3(R+r)^3(R-r)}$$

and the volume of the cone obtained by rotating OAU is

$$\frac{16\pi R^2 r^5}{3(R+r)^3(R-r)}$$

so that the volume of the frustum obtained by rotating $AUVB$ is

$$\frac{16\pi R^2r^2(R^3-r^3)}{3(R+r)^3(R-r)} = \frac{16\pi R^2r^2}{3(R+r)^3}(R^2+Rr+r^2) \quad .$$

The volume of a slice of a sphere of radius a and height h from the equatorial plane is

$$\pi\int_0^h (a^2-t^2)dt = \pi[a^2h - h^3/3] \ .$$

The portion of the larger sphere included within the frustum has volume

$$\begin{aligned}\frac{2\pi R^3}{3} - \pi\left[R^3\left(\frac{R-r}{R+r}\right) - \frac{R^3}{3}\left(\frac{R-r}{R+r}\right)^3\right] \\ = \frac{\pi R^3}{3}\left[2 - 3\left(\frac{R-r}{R+r}\right) + \left(\frac{R-r}{R+r}\right)^3\right] \\ = \frac{\pi R^3}{3(R+r)^3}[4r^3+12Rr^2] = \frac{4\pi R^2r^2}{3(R+r)^3}[Rr+3R^2]\end{aligned}$$

and the portion of the smaller sphere included within the frustum has volume

$$\frac{2\pi r^3}{3} + \pi\left[r^3\left(\frac{R-r}{R+r}\right) - \frac{r^3}{3}\left(\frac{R-r}{R+r}\right)^3\right] = \frac{4\pi R^2r^2}{3(R+r)^3}[Rr+3r^2] \quad .$$

Hence, the portions of the sphere lying within the frustum have total volume

$$\frac{4\pi R^2r^2}{3(R+r)^3}[3R^2+2Rr+3r^2] \ .$$

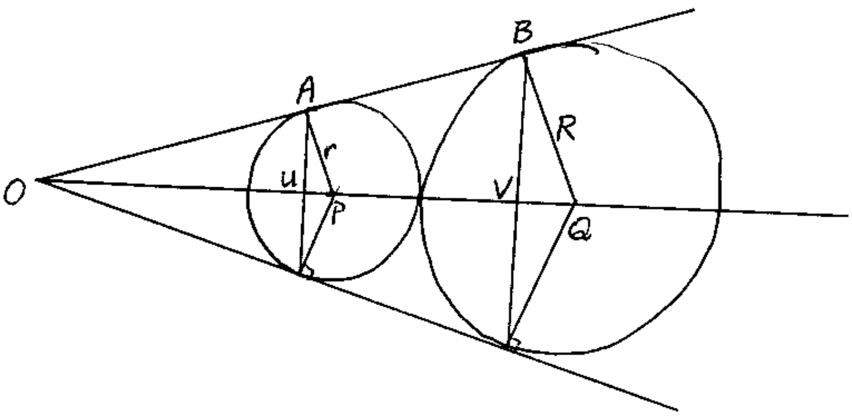
Subtracting this from the volume of the frustum yields the volume of the ring-shaped region

$$\frac{4\pi R^2r^2}{3(R+r)^3}[(4R^2+4Rr+4r^2)-(3R^2+2Rr+3r^2)] = \frac{4\pi R^2r^2}{3(R+r)^3}[R^2+2Rr+r^2] = \frac{4\pi R^2r^2}{3(R+r)} \quad .$$

Comment. The volume of a slice of a sphere of radius a and height h from the equatorial plane can be obtained from the volume of a right circular cone and a cylinder using the method of Cavalieri. The area of a cross-section of the slice at height t from the equator is $\pi(a^2 - t^2) = \pi a^2 - \pi t^2$. The term πa^2 represents the cross-section of a cylinder of radius a and height h while πt^2 represents the area of the cross section of a cone of base radius h at distance t from the vertex. Thus the area of the each cross-section of the cylinder is the sum of the areas of the corresponding cross-sections of the spherical slice and cone. Cavalieri's principle says that the volumes of the solids bear the same relation. Thus the volume of the spherical slice is

$$\pi a^2 h - \frac{1}{3}\pi h^3 \quad .$$

23.



24.

24. *First solution.* Suppose AD splits BC into segments of lengths a_1 and a_2 as in the diagram and angle BDC into subangles α_1 and α_2, and let $b_1, b_2, \beta_1, \beta_2, c_1, c_2, \gamma_1, \gamma_2$ be similarly defined as in the diagram.

Applying the Law of Sines to ΔBPD and ΔCPD, we find that

$$\frac{a_1}{\sin \alpha_1} = \frac{a_2}{\sin \alpha_2}$$

and similarly that

$$\frac{b_1}{\sin \beta_1} = \frac{b_2}{\sin \beta_2} \quad \text{and} \quad \frac{c_1}{\sin \gamma_1} = \frac{c_2}{\sin \gamma_2} \quad .$$

Let $\alpha = \angle BAE$. Then $\alpha = \angle FAC$ since $\angle FAB = \angle EAC$. Similarly, let $\beta = \angle FBC = \angle ABD$ and $\gamma = \angle BCE = \angle ACD$.

Let $|AB| = c$, $|BC| = a$, $|AC| = b$, $|AD| = u$, $|BE| = v$, $|CF| = w$. By the Law of Sines, we find that

$$\frac{v}{\sin \alpha} = \frac{c}{\sin \beta_1} \quad \text{and} \quad \frac{v}{\sin \gamma} = \frac{a}{\sin \beta_2}$$

so that

$$\frac{c \sin \alpha}{\sin \beta_1} = \frac{a \sin \gamma}{\sin \beta_2} \Longrightarrow \frac{\sin \beta_1}{\sin \beta_2} = \frac{c}{a} \cdot \frac{\sin \alpha}{\sin \gamma} \quad .$$

Similarly

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{b}{c} \cdot \frac{\sin \gamma}{\sin \beta} \quad \text{and} \quad \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{a}{b} \cdot \frac{\sin \beta}{\sin \alpha} \quad .$$

Putting this altogether yields

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = \frac{\sin \alpha_1}{\sin \alpha_2} \cdot \frac{\sin \beta_1}{\sin \beta_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{b}{c} \cdot \frac{c}{a} \cdot \frac{a}{b} \cdot \frac{\sin \gamma}{\sin \beta} \cdot \frac{\sin \alpha}{\sin \gamma} \cdot \frac{\sin \beta}{\sin \alpha} = 1 \ .$$

By the converse of Ceva's Theorem, the cevians AP, BQ and CR are concurrent and the result follows.

24. First solution