Mathematical Olympiads' Correspondence Program - 1997-1998

Problem Set 3

13. ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The right bisector of BC meets DE at P. Prove that $\angle BPC = \angle BAC$.
(*Note:* This is Problem 2244 in $CM + MM$, May, 1997.)

14. (a) Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

15. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number τ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer n, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer n, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let G_n be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and F_n^2. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

16. (a) Show that, if

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \quad ,$$

then

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1 \quad .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

17. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \quad , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

18. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

13.

13. *First solution.* Let the lengths a, b, c, u and the angles α, β, γ, λ, μ, ν be as indicated in the diagram.

In the solution, we make use of the fact that if $p/q = r/s$, then both fractions are equal to $(p+r)/(q+s)$. Since $\angle DBP = 90° + \lambda - 2\beta$, it follows that

$$2\mu = 180° - (90° - \alpha) - (90° + \lambda - 2\beta) = \alpha + 2\beta - \lambda \quad .$$

Similarly, $2\nu = \alpha + 2\gamma - \lambda$. Using the Law of Sines, we find that

$$\frac{a}{\sin 2\alpha} = \frac{b}{\sin 2\beta} = \frac{c}{\sin 2\gamma} = \frac{b+c}{\sin 2\beta + \sin 2\gamma} = \frac{b+c}{2\sin(\beta+\gamma)\cos(\beta-\gamma)}$$
$$= \frac{b+c}{2\cos\alpha\cos(\beta-\gamma)} \ .$$

Hence

$$\frac{a}{\sin\alpha} = \frac{b+c}{\cos(\beta-\gamma)} \ .$$

Since $a = 2u\sin\lambda$ and, by the Law of Sines,

$$\frac{u}{\sin(90^\circ - \alpha)} = \frac{b}{\sin 2\mu} \quad \text{and} \quad \frac{u}{\sin(90^\circ-\alpha)} = \frac{c}{\sin 2\nu} \ ,$$

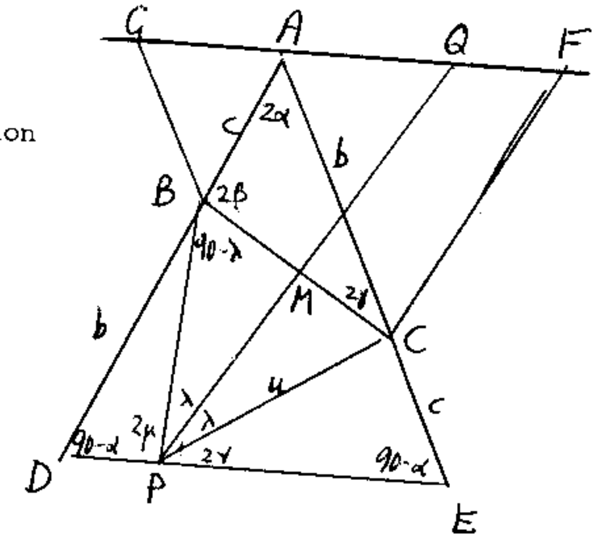
we have that

$$\frac{a}{2\sin\lambda\cos\alpha} = \frac{u}{\cos\alpha} = \frac{b}{\sin 2\mu} = \frac{c}{\sin 2\nu} = \frac{b+c}{\sin 2\mu + \sin 2\nu}$$
$$= \frac{b+c}{2\sin(\mu+\nu)\cos(\mu-\nu)} = \frac{b+c}{2\cos\lambda\cos(\beta-\gamma)} = \frac{a}{2\cos\lambda\sin\alpha} \ .$$

Hence $\tan\alpha = \tan\lambda$ and so $\alpha = \lambda$.

13.

First solution



13. *Second solution.* Let M be the midpoint of BC. A rotation of 180° about M interchanges B and C and takes E to G, D to F and P to Q. Then $AB = CE = BG$ and $AC = BD = CF$. Join GA and FA. Let $2\alpha = \angle BAC$. Since $AE \| BG$ and AB is a transversal, $\angle GBA = \angle BAC = 2\alpha$. Since $AB = BG$, $\angle BGA = 90^\circ - \alpha$. But $\angle BGF = \angle CED = 90^\circ - \alpha$. Thus, G, A, F are collinear.

Since GF and DE are equidistant from M, we can use Cartesian coordinates with the origin at M, the line $y = 1$ as GF and the line $y = -1$ as DE. Let $A \sim (a, 1)$, $B \sim (-u, -mu)$, $C \sim (u, mu)$. Then $P \sim (m, -1)$, $Q \sim (-m, 1)$,

$$D \sim (a - \frac{2(a+u)}{1+mu}, -1), \qquad E \sim (a + \frac{2(a+u)}{1+mu}, -1) \ .$$

Since $|AC| = |BD|$, we find that $u - a = -u - a + \frac{2(a+u)}{1+mu}$, or $a = mu^2$. (We can check this by equating the slopes of AC and AE.)

The slope of AE is $-1/u$ and of AD is $1/u$, so that

$$\tan \angle BAC = \frac{-(2/u)}{1-(1/u^2)} = -\frac{2u}{u^2-1} \ .$$

The slope of CQ is $(mu-1)/(m+u)$ and of BQ is $(1+mu)/(u-m)$, so that

$$\begin{aligned}\tan \angle BPC = \tan \angle BQC &= \frac{(mu-1)(u-m)-(mu+1)(u+m)}{(u-m)(u+m)+(mu-1)(mu+1)} \\ &= \frac{-2(m^2u+u)}{u^2-m^2+m^2u^2-1} = \frac{-2(m^2+1)u}{(1+m^2)(u^2-1)} = \frac{-2u}{u^2-1} \ .\end{aligned}$$

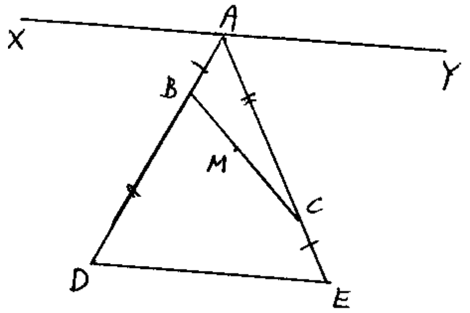
The result follows.

13. *Third solution.* [M. Boase] Let XAY be drawn parallel to DE.

Since M is the midpoint of BC, the distance from M to DE is the average of the distances from B and C to DE. Similarly, the distance from M to XY is the average of the distances from B and C to XY. The distance of B (resp. C) to DE equals the distance of C (resp. B) to XY. Hence, M is equidistant from DE and XY. If PM produced meets XY in Q, then $PM = MQ$ and so $\angle BQC = \angle BPC$.

Select R on MQ (possibly produced) so that $\angle BAC = \angle BRC$. Since $\Delta ADE ||| \Delta RBC$, $\angle RBC = \angle RCB = \angle ADE$. Since $BARC$ is a concyclic quadrilateral, $\angle BAR = 180° - \angle RCB = 180° - \angle ADE = 180° - \angle XAD = \angle BAQ$ from which it follows that $R = Q$ and so $\angle BPC = \angle BQC = \angle BRC = \angle BAC$.

Third solution



13. *Fourth solution.* [Jimmy Chui] Set coordinates: $A \sim (0, (m+n)b)$, $B \sim (-ma, nb)$, $C \sim (na, mb)$ $D \sim (-(m+n)a, 0)$ and $E \sim ((m+n)a, 0)$ where $m = |AB|$, $n = |AC|$ and $a^2 + b^2 = 1$. Then the line BC has the equation

$$\frac{m-n}{a}x - \frac{m+n}{b}y + m^2 + n^2 = 0$$

and the right bisector of BC has equation

$$\frac{m+n}{b}x + \frac{m-n}{a}y + \frac{(a^2-b^2)(m^2-n^2)}{2ab} = 0 \ .$$

Thus

$$P \sim \left(\frac{(b^2-a^2)(m-n)}{2a}, 0 \right) \ .$$

Now

$$|BC|^2 = m^2 + n^2 + 2mn(a^2 - b^2)$$

and

$$|BP|^2 = \frac{m^2 + n^2 + 2mn(a^2 - b^2)}{4a^2}$$

so that $|BC|/|BP| = 2a$. Also $|DE|/|AD| = 2(m+n)a/(m+n) = 2a$ so that ΔBPC is similar to ΔADE and the result follows.

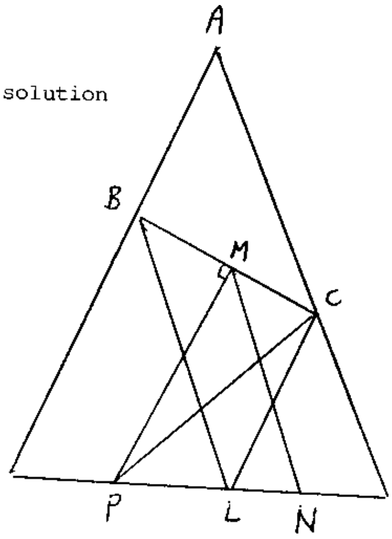
13. *Fifth solution.* Determine points L and N on DE such that $BL \| AE$ and $LN = NE$. Now

$$\frac{LE}{LD} = \frac{AB}{BD} = \frac{CE}{CA}$$

so that $CL \| AD$ and $CL : AD = CE : AE$. Since $AD = DE$, $CL = CE$ and so $CN \perp LE$. Consider the trapezoid $CBLE$. The line MN joins the midpoints of the nonparallel opposite sides and so $MN \| BL$. $MPNC$ is a quadrilateral with right angles at M and N, and so is concyclic. Hence

$$\angle BPC = 2\angle MPC = 2\angle MNC = 2\angle NCE = \angle LCE = \angle BAC \quad .$$

Fifth solution



13. *Sixth solution.* [C. So] Let F, N, G be the feet of the perpendiculars dropped from B, M, C respectively to DE. Note that $FN = NG$, so that $MF = MG$. Let $\angle ADE = \angle AED = \theta$, $|AB| = c$, $|AC| = b$ and h be the altitude of ΔADE. Then

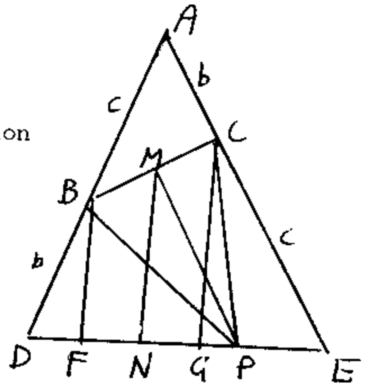
$$|MN| = \frac{1}{2}[|BF| + |CG|] = \frac{1}{2}(b+c)\sin\theta = \frac{h}{2}$$

and

$$|DF| = b\cos\theta \ , \quad |GE| = c\cos\theta \ , \quad |DE| = 2(b+c)\cos\theta \quad .$$

Hence $|FG| = |DE| - |DF| - |GE| = \frac{1}{2}|DE|$. Since ΔADE and ΔMFG are isosceles triangles with heights and beses in proportion, they are similar so that $\angle MFG = \angle ADE = \theta$. Since $\angle BFP = \angle BMG = 90°$, the quadrilateral $BFPM$ is concyclic and so $\angle CBP = \angle MFP = \theta$ (we are supposing that the configuration is labelled so P lies between F and E). Hence ΔADE is similar to ΔPCB and so $\angle BPC = \angle BAC$.

Sixth solution



13. *Seventh solution.* [A. Chan] Let $\angle ADE = \angle AED = \theta$, so $\angle BAC = 180° - 2\theta$. Suppose that $\angle ACB = \phi$, $\angle CPE = \sigma$ and $\angle BCP = \rho$. By the Law of Sines for triangles ABC and PCE, we find that

$$\frac{2|PC|\cos\rho}{\sin 2\theta} = \frac{|AB|}{\sin\phi}$$

whence

$$\frac{\sin\sigma}{\sin\theta} = \frac{|CE|}{|PC|} = \frac{|AB|}{|PC|} = \frac{2\cos\rho\sin\phi}{\sin 2\theta}$$

and

$$\sin\sigma\cos\theta = \sin\phi\cos\rho \quad .$$

Therefore

$$\sin(\theta+\sigma) + \sin(\sigma-\theta) = \sin(\phi+\rho) + \sin(\phi-\rho).$$

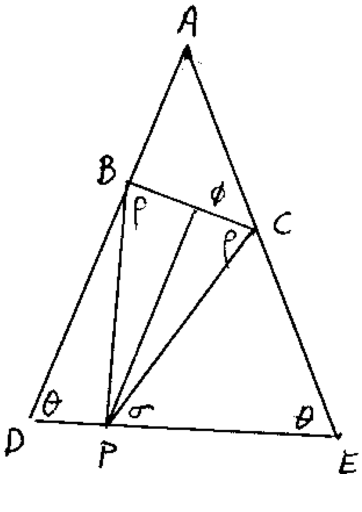
Since $\theta+\sigma = \phi+\rho$, $\sin(\sigma-\theta) = \sin(\phi-\rho)$. Either $(\sigma-\theta)+(\phi-\rho) = \pm 180°$ or $\sigma-\theta = \phi-\rho$. In the first case, since $\theta+\sigma = \phi+\rho$, $|\sigma-\rho| = 90°$, which is false.

Hence $\sigma-\theta = \phi-\rho$, so, with $\theta+\sigma = \phi+\rho$, we have that

$$2\theta = \theta + (\rho+\sigma-\phi) = \theta + (\rho+\rho-\sigma) = 2\rho$$

and the result follows.

Seventh solution



14.

14. (a) *First solution.* Since $c = a + d - b$, we have that

$$bc - ad = b(a + d - b) - ad = (a - b)b - (a - b)d = (a - b)(b - d) > 0 \quad .$$

(a) *Second solution.* Let $a + d = b + c = u$. Then

$$bc - ad = b(u - b) - (u - d)d = u(b - d) - (b^2 - d^2) = (b - d)(u - b - d) \ .$$

Now $u = b + c > b + d$, so that $u - b - d > 0$ as well as $b - d > 0$. Hence $bc - ad > 0$ as desired.

(a) *Third solution.* Let $x = a - b > 0$. Since $a - b = c - d$, we have that $a = b + x$ and $d = c - x$. Hence

$$bc - ad = bc - (b + x)(c - x) = bx - cx + x^2 = x^2 + x(b - c) > 0 \ .$$

(a) *Fourth solution.* Since $\sqrt{a} > \sqrt{b} > \sqrt{c} > \sqrt{d}$, $\sqrt{a} - \sqrt{d} > \sqrt{b} - \sqrt{c}$. Squaring and using $a + d = b + c$ yields $2\sqrt{bc} > 2\sqrt{ad}$, whence the result.

14. (b) *First solution.* Since all variables represent integers,

$$aq - bp > 0, br - as > 0 \Longrightarrow aq - bp \geq 1, br - as \geq 1 \ .$$

Therefore

$$b = b(qr - ps) = q(br - as) + s(aq - bp) \geq q + s \ .$$

15.

15. (a) *First solution.* Since $F_0 = F_2 - F_1 = 0$, the result holds for $n = 0$. Since $F_{-1} = F_1 - F_0 = 1$, the result holds for $n = 1$. Suppose that we have established the result for $n = 0, 1, 2, \cdots r$. Then

$$F_{-(r+1)} = F_{-r-1} = F_{-r+1} - F_{-r} = (-1)^r F_{r-1} - (-1)^{r+1} F_r = (-1)^{r+2}(F_{r-1} + F_r) = (-1)^{r+2} F_{r+1} \ .$$

The result follows by induction.

15. (b) *First solution.* The result holds for $n = 0$, $n = 1$ and $n = 2$. Suppose that it holds for $n = 0, 1, 2, \cdots, r$. Then

$$\tau^{r+1} = \tau^r + \tau^{r-1} = (F_r + F_{r-1})\tau + (F_{r-1} + F_{r-2}) = F_{r+1}\tau + F_r\tau \ .$$

This establishes the result for positive values of n. Now $\tau^{-1} = \tau - 1 = F_{-1}\tau + F_{-2}$, so the result holds for $n = -1$. Suppose that we have established the result for $n = 0, -1, -2, \cdots, -r$. Then

$$\tau^{-(r+1)} = \tau^{-(r-1)} - \tau^{-r} = (F_{-(r-1)} - F_{-r})\tau + (F_{-r} - F_{-(r+1)}) = F_{-(r+1)}\tau + F_{-(r+2)} \ .$$

15. (b) *Second solution.* The result holds for $n = 1$. Suppose that it holds for $n = r \geq 0$. Then

$$\tau^{r+1} = \tau^r \cdot \tau = (F_r\tau + F_{r-1})\tau = F_r\tau^2 + F_{r-1}\tau$$

$$= (F_r + F_{r-1})\tau + F_r = F_{r+1}\tau + F_r \quad .$$

Now consider nonpositive values of n. We have that $\tau^0 = 1$, $\tau^{-1} = \tau - 1$, $\tau^{-2} = 1 - \tau^{-1} = 2 - \tau$. Suppose that we have shown for $r \geq 0$ that $\tau^{-r} = F_{-r}\tau + F_{-r-1}$. Then

$$\tau^{-(r+1)} = \tau^{-1}\tau^{-r} = F_{-r} + F_{-r-1}(\tau - 1) = F_{-r-1}\tau + (F_{-r} - F_{-r-1})$$

$$= F_{-r-1}\tau + F_{-r-2} = F_{-(r+1)}\tau + F_{-(r+1)-1} \quad .$$

By induction, it follows that the result holds for both positive and negative values of n.

15. (c) *First solution.* Let $G_n = F_n F_{n+1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+3} + F_{n+1}F_n \\ &= (F_{n+3} + F_{n+2})(F_{n+2} + F_{n+1}) + (F_{n+3} - F_{n+2})(F_{n+2} - F_{n+1}) \\ &= 2(F_{n+3}F_{n+2} + F_{n+2}F_{n+1}) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_{n+1}F_{n-1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+2} + F_{n+1}F_{n-1} \\ &= (F_{n+3} + F_{n+2})(F_{n+1} + F_n) + (F_{n+3} - F_{n+2})(F_{n+1} - F_n) \\ &= 2(F_{n+3}F_{n+1} + F_{n+2}F_n) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_n^2$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+3}^2 + F_n^2 = (F_{n+2} + F_{n+1})^2 + (F_{n+2} - F_{n+1})^2 \\ &= F_{n+2}^2 + 2F_{n+2}F_{n+1} + F_{n+1}^2 + F_{n+2}^2 - 2F_{n+2}F_{n+1} + F_{n+1}^2 = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Comments. Since $F_n^2 = F_nF_{n-1} + F_nF_{n-2}$, the third result of (c) can be obtained from the first two. J. Chui observed that, more generally, we can take $G_n = F_{n+u}F_{n+v}$ where u and v are integers. Then

$$\begin{aligned} &G_{n+3} + G_n - 2(G_{n+1} + G_{n+2}) \\ &= (F_{n+3+u}F_{n+3+v} + F_{n+u}F_{n+v}) - 2(F_{n+2+u}F_{n+2+v} + F_{n+1+u}F_{n+1+v}) \\ &= (2F_{n+1+u} + F_{n+u})(2F_{n+1+v} + F_{n+v}) + F_{n+u}F_{n+v} \\ &\qquad - 2(F_{n+1+u} + F_{n+u})(F_{n+1+v} + F_{n+v}) - 2F_{n+1+u}F_{n+1+v} \\ &= 0 \ , \end{aligned}$$

so that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

16.

16. (a) *First solution.* Let

$$\lambda = \frac{\cos\beta}{\cos\alpha} \quad \text{and} \quad \mu = \frac{\sin\beta}{\sin\alpha} \ .$$

Since $\lambda^{-1} + \mu^{-1} = -1$, we have that $\lambda + \mu = -\lambda\mu$. Now

$$1 = \cos^2\beta + \sin^2\beta = \lambda^2\cos^2\alpha + \mu^2\sin^2\alpha = \lambda^2 + (\mu^2 - \lambda^2)\sin^2\alpha = \lambda^2 - (\mu - \lambda)\lambda\mu\sin^2\alpha \ .$$

Hence

$$\begin{aligned} \frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \lambda^3\cos^2\alpha + \mu^3\sin^2\alpha \\ &= \lambda(\lambda^2\cos^2\alpha + \mu^2\sin^2\alpha) + (\mu - \lambda)\mu^2\sin^2\alpha \\ &= \lambda + (\mu - \lambda)\mu^2\sin^2\alpha \\ &= \frac{1}{\lambda}[\lambda^2 + (\lambda^2 - 1)\mu] \\ &= \frac{1}{\lambda}[\lambda^2 + \lambda^2\mu + \lambda + \lambda\mu \\ &= \lambda + \lambda\mu + 1 + \mu = 1 \ . \end{aligned}$$

16. (a) *Second solution.* [M. Boase]

$$\frac{\cos\alpha}{\cos\beta}+\frac{\sin\alpha}{\sin\beta}=-1\Longrightarrow$$
$$\sin(\alpha+\beta)+\sin\beta\cos\beta=0\ . \qquad (*)$$

Therefore

$$\begin{aligned}\frac{\cos^3\beta}{\cos\alpha}+\frac{\sin^3\beta}{\sin\alpha}&=\frac{\cos\beta(1-\sin^2\beta)}{\cos\alpha}+\frac{\sin\beta(1-\cos^2\beta)}{\sin\alpha}\\&=\frac{\cos\beta}{\cos\alpha}+\frac{\sin\beta}{\sin\alpha}-\sin\beta\cos\beta\left(\frac{\sin\beta}{\cos\alpha}+\frac{\cos\beta}{\sin\alpha}\right)\\&=\frac{\sin(\alpha+\beta)}{\cos\alpha\sin\alpha}-\frac{\cos\beta\sin\beta(\cos(\alpha-\beta))}{\cos\alpha\sin\alpha}\\&=\frac{-2\sin\beta\cos\beta+2\sin(\alpha+\beta)\cos(\alpha-\beta)}{2\sin\alpha\cos\alpha}\quad\text{using }\ (*)\\&=\frac{-2\sin\beta\cos\beta+[\sin2\alpha+\sin2\beta]}{\sin2\alpha}=1\end{aligned}$$

since $2\sin\beta\cos\beta=\sin2\beta$.

16. (a) *Third solution.* [A. Birka] Let $\cos\alpha=x$ and $\cos\beta=y$. Then

$$\frac{\sin\alpha}{\sin\beta}=\pm\sqrt{\frac{1-x^2}{1-y^2}}\ .$$

Since

$$\frac{x}{y}+1=\mp\sqrt{\frac{1-x^2}{1-y^2}}\ .$$

then

$$(x^2+2xy+y^2)(1-y^2)=y^2(1-x^2)\ ,$$

whence

$$x^2+2xy=2xy^3+y^4\ .$$

Thus,

$$\begin{aligned}\frac{\cos^3\beta}{\cos\alpha}+\frac{\sin^3\beta}{\sin\alpha}&=\frac{y^3}{x}\pm(1-y^2)\sqrt{\frac{1-y^2}{1-x^2}}\\&=\frac{y^3}{x}-\frac{(1-y^2)y}{x+y}=\frac{y^4+2xy^3-xy}{x(x+y)}\\&=\frac{x^2+xy}{x(x+y)}=1\ .\end{aligned}$$

16. (a) *Fourth solution.* [J. Chui] Note that the given equation implies that $\sin2\beta=-2\sin(\alpha+\beta)$ and that the numerator of

$$\frac{\cos\alpha}{\cos\beta}+\frac{\sin\alpha}{\sin\beta}+\frac{\cos^3\beta}{\cos\alpha}+\frac{\sin^3\beta}{\sin\alpha}$$

is one quarter of

$$\begin{aligned}&4[\cos^2\alpha\sin\alpha\sin\beta+\sin^2\alpha\cos\alpha\cos\beta+\cos^4\beta\sin\alpha\sin\beta+\sin^4\beta\cos\alpha\cos\beta]\\&\quad=4[\cos^2\alpha\sin\alpha\sin\beta+\sin^2\alpha\cos\alpha\cos\beta+(\cos^2\beta-\cos^2\beta\sin^2\beta)\sin\alpha\sin\beta\\&\qquad+(\sin^2\beta-\sin^2\beta\cos^2\beta)\cos\alpha\cos\beta]\\&\quad=(4\cos^2\alpha+4\cos^2\beta-\sin^22\beta)\sin\alpha\sin\beta+(4\sin^2\alpha+4\sin^2\beta-\sin^22\beta)\cos\alpha\cos\beta\\&\quad=2\sin2\alpha\cos\alpha\sin\beta+2\sin2\beta\cos\beta\sin\alpha+2\sin2\alpha\sin\alpha\cos\beta+2\sin2\beta\cos\alpha\sin\beta\\&\qquad-\sin^22\beta(\cos\alpha\cos\beta+\sin\alpha\sin\beta)\\&\quad=2(\sin2\alpha+\sin2\beta)\sin(\alpha+\beta)-\sin^22\beta\cos(\alpha-\beta)\\&\quad=2\sin(\alpha+\beta)[\sin2\alpha+\sin2\beta-2\sin(\alpha+\beta)\cos(\alpha-\beta)]=0\ ,\end{aligned}$$

since

$$\sin 2\alpha + \sin 2\beta = \sin(\overline{\alpha+\beta} + \overline{\alpha-\beta}) + \sin(\overline{\alpha+\beta} - \overline{\alpha-\beta}) \ .$$

16. (a) *Fifth solution.* [A. Tang] From the given equation, we have that

$$\sin(\alpha+\beta) = -\sin\beta\cos\beta \quad ,$$

$$\frac{\cos\beta}{\cos\alpha} = \frac{-\sin\beta}{\sin\alpha+\sin\beta} \quad ,$$

and

$$\frac{\sin\beta}{\sin\alpha} = \frac{-\cos\beta}{\cos\alpha+\cos\beta} \quad .$$

Hence

$$\begin{aligned}\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \cos^2\beta\left[\frac{-\sin\beta}{\sin\alpha+\sin\beta}\right] + \sin^2\beta\left[\frac{-\cos\beta}{\cos\alpha+\cos\beta}\right] \\ &= -\frac{\sin\beta\cos\beta[\cos\alpha\cos\beta+\sin\alpha\sin\beta+1]}{4\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)} \\ &= \frac{\sin(\alpha+\beta)[\cos(\alpha-\beta)+1]}{[2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)][2\cos^2\frac{1}{2}(\alpha-\beta)]} = 1 \quad .\end{aligned}$$

16. (a) *Sixth solution.* [D. Arthur] The given equations yield $2\sin(\alpha+\beta) = -\sin 2\beta$, $\cos\alpha\sin\beta = -\cos\beta(\sin\alpha+\sin\beta)$ and $\sin\alpha\cos\beta = -\sin\beta(\cos\alpha+\cos\beta)$. Hence

$$\begin{aligned}\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos^2\beta(\cos\beta\sin\alpha)+\sin^2\beta(\sin\beta\cos\alpha)}{\cos\alpha\sin\alpha} \\ &= \frac{-\cos^2\beta\sin\beta(\cos\alpha+\cos\beta)-\sin^2\beta\cos\beta(\sin\alpha+\sin\beta)}{\cos\alpha\sin\alpha} \\ &= \frac{-\cos\beta\sin\beta(\cos\alpha\cos\beta+\cos^2\beta+\sin\alpha\sin\beta+\sin^2\beta)}{\cos\alpha\sin\alpha} \\ &= \frac{-\sin 2\beta(1+\cos(\alpha-\beta))}{\sin 2\alpha} \\ &= \frac{-\sin 2\beta + 2\sin(\alpha+\beta)\cos(\alpha-\beta)}{\sin 2\alpha} \\ &= \frac{-\sin 2\beta+\sin 2\alpha+\sin 2\beta}{\sin 2\alpha} = 1 \quad .\end{aligned}$$

16. (b) *First solution.* The given equation is equivalent to $2\sin(\alpha+\beta)+\sin 2\beta = 0$. Try $\beta = -45°$ so that $\sin(\alpha - 45°) = \frac{1}{2}$. We take $\alpha = 75°$. Now

$$\sin 75° = \sin(45°+30°) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right)$$

and

$$\cos 75° = \cos(45°+30°) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\right) \quad .$$

It is straightforward to check that both equations hold.

17.

17. *First solution.* Taking $x = 1, 2, 3, \cdots$ yields the respective sequences

$$\{1,1,0,-1,-1,0,\cdots\} \ , \quad \{2,8,30,112,418,1560,\cdots\} \ , \quad \{3,27,240,2133,\cdots\} \ .$$

In each case, we find that

$$P_{n+1}(x) = x^2 P_n(x) - P_{n-1}(x) \tag{1}$$

for $n = 2, 3, \cdots$. If we can establish (1) in general, it will follow that all the functions P_n are polynomials.

From the definition of P_n, we find that

$$P_{n+1} + P_{n-1} = P_n(P_n^2 - P_{n+1}P_{n-1}) \quad .$$

Therefore, it suffices to establish that $P_n^2 - P_{n+1}P_{n-1} = x^2$ for each n. Now, for $n \geq 2$,

$$\begin{aligned}
[P_{n+1}^2 - P_{n+2}P_n] - [P_n^2 - P_{n+1}P_{n-1}] &= P_{n+1}(P_{n+1} + P_{n-1}) - P_n(P_{n+2} + P_n) \\
&= P_{n+1}P_n(P_n^2 - P_{n+1}P_{n-1}) - P_nP_{n+1}(P_{n+1}^2 - P_{n+2}P_n) \\
&= -P_{n+1}P_n[(P_{n+1}^2 - P_{n+2}P_n) - (P_n^2 - P_{n+1}P_{n-1})] \quad ,
\end{aligned}$$

so that either $P_{n+1}(x)P_n(x) + 1 \equiv 0$ or $P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$. The first identity is precluded by the case $x = 1$, where it is false. Hence

$$P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$$

for $n = 2, 3, \cdots$. Since $P_2^2(x) - P_3(x)P_1(x) = x^2$, the result follows.

17. *Second solution.* [By inspection, we make the conjecture that $P_n(x) = x^2P_{n-1}(x) - P_{n-2}$. Rather than prove this directly from the rather awkward condition on P_n, we go through the back door.] Define the sequence $\{Q_n\}$ for $n = 0, 1, 2, \cdots$ by

$$Q_0(x) = 0 \ , \qquad Q_1(x) = x \ , \qquad Q_{n+1} = x^2Q_n(x) - Q_{n-1}(x)$$

for $n \geq 1$. It is clear that $Q_n(x)$ is a polynomial of degree $2n - 1$ for $n = 1, 2, \cdots$. We show that $P_n(x) = Q_n(x)$ for each n.

Lemma: $Q_n^2(x) - Q_{n+1}Q_{n-1} = x^2$ for $n \geq 1$.

Proof: This result holds for $n = 1$. Assume that it holds for $n = k - 1 \geq 1$. Then

$$\begin{aligned}
Q_k^2(x) - Q_{k+1}(x)Q_{k-1}(x) &= Q_k^2(x) - (x^2Q_k(x) - Q_{k-1}(x))Q_{k-1}(x) \\
&= Q_k(x)(Q_k(x) - x^2Q_{k-1}(x)) + Q_{k-1}^2(x) \\
&= -Q_k(x)Q_{k-2}(x) + Q_{k-1}^2(x) = x^2 \ . \qquad \textbf{QED}
\end{aligned}$$

From the lemma, we find that

$$\begin{aligned}
&Q_{n+1}(x) + Q_{n-1}(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) \\
&\quad = x^2Q_n(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) = Q_n(x)(x^2 + Q_{n+1}(x)Q_{n-1}(x)) = Q_n^3(x)
\end{aligned}$$

$$\Longrightarrow Q_{n+1}(x) = \frac{Q_n^3(x) - Q_{n-1}(x)}{1 + Q_n(x)Q_{n-1}(x)} \qquad (n = 1, 2, \cdots) \ .$$

We know that $Q_1(x) = P_1(x)$ and $Q_2(x) = P_2(x)$. Suppose that $Q_n(x) = P_n(x)$ for $n = 1, 2, \cdots, k$. Then

$$Q_{k+1}(x) = \frac{Q_k^3(x) - Q_{k-1}(x)}{1 + Q_k(x)Q_{k-1}(x)} = \frac{P_k^3(x) - P_{k-1}(x)}{1 + P_k(x)P_{k-1}(x)} = P_{k+1}(x)$$

from the definition of P_{k+1}. The result follows.

Comment: It can also be established that $P_{n+1}^2 + P_n^2 = (1 + P_nP_{n+1})x^2$ for each $n \geq 0$.

17. *Third solution.* [I. Panayotov] First note that the sequence $\{P_n(x)\}$ is defined for all values of x, *i.e.*, the denominator $1 + P_{n-1}(x)P_n(x)$ never vanishes for n and x. Suppose otherwise, and let n be the least number for which there exists u for which $1 + P_{n-1}(u)P_n(u) = 0$. Then $n \geq 3$ and

$$-1 = P_{n-1}(u)P_n(u) = \frac{P_{n-1}(u)^4 - P_{n-1}(u)P_{n-2}(u)}{1 + P_{n-1}(u)P_{n-2}(u)}$$

which implies that $P_{n-1}(u)^4 = -1$, a contradiction.

We now prove by induction that $P_{n+1} = x^2P_n - P_{n-1}$. Suppose that $P_k = x^2P_{k-1} - P_{k-2}$ for $3 \leq k \leq n$, so that in particular we know that P_k is a polynomial for $1 \leq k \leq n$. Substituting for P_k yields

$$P_{k-1}^3(x) = P_{k-1}(x)[x^2 + x^2P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x)]$$

for all x. If $P_{k-1}(x) \neq 0$, then

$$P_{k-1}^2(x) = x^2 + x^2P_{k-1}P_{k-2}(x) - P_{k-2}^2(x) \ .$$

Both sides of this equation are polynomials and so continuous functions of x. Since the roots of P_{k-1} constitute a finite discreet set, this equation holds when x is one of the roots as well. Now

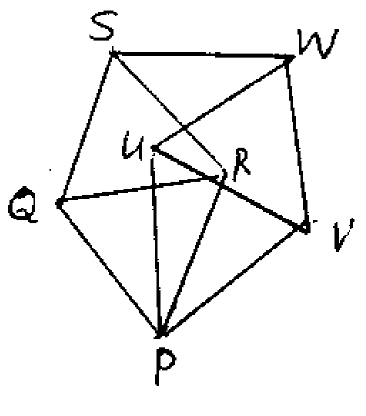
$$\begin{aligned} P_{n+1} &= \frac{P_n^3 - P_{n-1}}{1 + P_nP_{n-1}} = \frac{P_n(x^2P_{n-1} - P_{n-2})^2 - P_{n-1}}{1 + P_nP_{n-1}} \\ &= \frac{P_n(x^4P_{n-1}^2 - x^2P_{n-1}P_{n-2} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_nP_{n-1}} \\ &= \frac{P_n(x^2P_nP_{n-1} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_nP_{n-1}} \qquad \text{since} \quad x^2P_{n-1} - P_{n-2} = P_n \\ &= \frac{(x^2P_n - P_{n-1})(1 + P_nP_{n-1})}{1 + P_nP_{n-1}} = x^2P_n - P_{n-1} \ . \end{aligned}$$

The result follows.

18.

18. *First solution.* Suppose that the points in the plane are coloured with three colours. Select any point P.

Consider the diagrammed configuration, in which all segments have length 1. If P, Q, R are all coloured differently, then either the result holds or S must have the same colour as P. If P, U, V are all coloured differently, then either the result holds or W must have the same colour as P. Hence, either one of the triangles PQR and PUV has two vertices the same colour, or else S and W must be coloured the same.



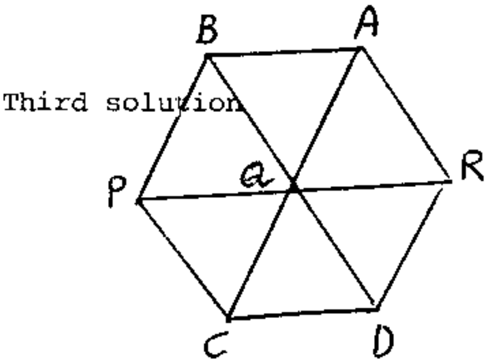
First solution

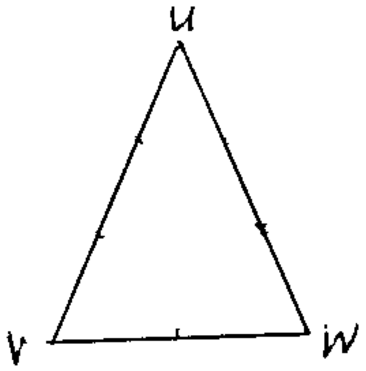
18. *Second solution.* Suppose, if possible, the planar points can be coloured without two points unit distance apart being coloured the same. Then if A and B are distant $\sqrt{3}$ apart, then there are distinct points C and D such that ACD and BCD are equilateral triangles ($ABCD$ is a rhombus). Since A and B must be coloured differently from the two colours of C and D, A and B must have the same colour. Hence, if O is any point in the plane, every point on the circle of radius $\sqrt{3}$ consists of points coloured the same as O. But there are two points on this circle unit distant apart, and we get a contradiction of our initial assumption.

18. *Third solution.* Suppose we can colour the points of the plane with three colours, red, blue and yellow so that the result fails. We show that three collinear points at unit distance are coloured with three different colours. Let P, Q, R be three such points, and embed P, Q, R in the hexagon $ABPCDR$ as indicated with all indicated segments of unit length.

If, say, Q is red, B and A must be coloured differently, as are A and R, R and D, D and C, C and P, P and B. Thus, B, R, C, are one colour, say, blue, and A, D, P the other, say yellow. The preliminary result follows.

Now consider any isosceles triangle UVW with $|UV| = |UW| = 3$ and $|VW| = 2$. It follows from the preliminary result that U and V must have the same colour, as do U and W. But V and W cannot have the same colour and we reach a contradiction.





18. *Fourth solution.* [D. Arthur] Suppose that the result is false. Let A, B be two points with $|AB| = 3$. Within the segment AB select PQ with $|AP| = |PQ| = |QB| = 1$, and suppose that R and S are points on the same side of AB with ΔRAP and ΔSPQ equilateral. Then $|RS| = 1$. Suppose if possible that A and Q have the same colour. Then P must have a second colour and R and S the third, leading to a contradiction. Hence A must be coloured differently from both P and Q. Similarly B must be coloured differently from both P and Q. Since P and Q are coloured differently, A and B must have the same colour.

Now consider a trapezoid $ABCD$ with $|CB| = |AB| = |AD| = 3$ and $|CD| = 1$. By the foregoing observation, C, A, B, D must have the same colour. But this yields a contradiction. The result follows.