Mathematical Olympiads' Correspondence Program - 1997-1998
Problem Set 2

7. For each of the following expressions independently, determine as many integer values of x as you can that it is a perfect square. Indicate whether your list is complete or not.

   (a) $1 + x$;
   (b) $1 + x + x^2$;
   (c) $1 + x + x^2 + x^3$;
   (d) $1 + x + x^2 + x^3 + x^4$;
   (e) $1 + x + x^2 + x^3 + x^4 + x^5$.

8. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that
$$\frac{BD}{DC} \le \frac{BF}{FA} \le 1$$
and
$$\frac{AE}{EC} \le \frac{AF}{FB} \quad .$$
Prove that $[DEF] \le \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ. This is Problem 2240 in $CM + M^2$, May, 1997.)

9. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and A', B', C', D' be points on l such that AA', BB', CC', DD' are all perpendicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.
(*Note:* XY here and in the previous problem denotes the length of the segment XY; the dot is ordinary multiplication. This is Problem 2237 in $CM + MM$, April, 1997.)

10. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

    (b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

11. The faces of a tetrahedron are formed by congruent triangles. If α is the angle between a pair of opposite edges of the tetrahedron, show that
$$\cos\alpha = \frac{\sin(B - C)}{\sin(B + C)}$$
where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.

12. (a) Give an example of a pair a, b of positive integers, not both prime, for which $2a - 1$, $2b - 1$ and $a + b$ are all primes. Determine all possibilities for which a and b are themselves prime.

    (b) Suppose a and b are positive integers such that $2a - 1$, $2b - 1$ and $a + b$ are all primes. Prove that neither $a^b + b^a$ nor $a^a + b^b$ are multiples of $a + b$.

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

7.

7. (a) $1 + x$ is a square when $x = u^2 - 1$ for some integer u (or when x is the product of two integers $u - 1$ and $u + 1$ that differ by 2).

7. (b) *First solution.* Suppose that $x^2+x+1=u^2$. Then $(2x+1)^2+3=4x^2+4x+4=4u^2=(2u)^2$, whence

$$3=(2u)^2-(2x+1)^2=(2u+2x+1)(2u-2x-1)\ .$$

The factors on the right must be ± 3 and ± 1 in some order, and this leads to the possibilities $(x,u)=(-1,\pm 1),(0,\pm 1)$.

(b) *Second solution.* If $x>0$, then $x^2<x^2+x+1<(x+1)^2$, so that x^2+x+1 cannot be square. If $x<-1$, then $x^2>x^2+x+1>(x+1)^2$ and x^2+x+1 cannot be square. This leaves only the possibilities $x=0,-1$.

(b) *Third solution.* For given u, consider the quadratic equation

$$x^2+x+1=u^2\ .$$

Its discriminant is $1-4(1-u^2)=4u^2-3$. It will have integer solutions only if $4u^2-3=v^2$ for some integer v, *i.e.*, $(v+2u)(v-2u)=-3$. The only possibilities are $(u,v)=(\pm 1,\pm 1),(\pm 1,\mp 1)$.

(b) *Fourth solution.* [J. Chui] If $f(x)=1+x+x^2$, then $f(x)=f(-(x+1))$, so we need deal only with nonnegative values of x. We have that $f(0)=f(-1)=1$ is square. Let $x\geq 1$ and suppose that $1+x+x^2=u^2$ for some integer u. Then $(1+x)^2-u^2=x>0$ so that $1+x>u$. This implies that $x\geq u$, whence $x^2\geq u^2=x^2+x+1$, a contradiction. Thus the only possibilities are $x=0,-1$.

(b) *Fifth solution.* [A. Birka] Suppose that $x^2+x+1=u^2$ with $u\geq 0$. This is equivalent to $x=(1+x)^2-u^2=(1+x+u)(1+x-u)$, so that $1+x+u$ and $1+x-u$ both divide x. If $x\geq 1$, then $1+x+u$ exceeds x and so cannot divide x. If $x\leq 0$, then $(-x)+u-1$ divides x, which is impossible unless $u=1$ or $u=0$. Only $u=1$ is viable, and this leads to $x=0,-1$.

7. (c) *First solution.* $1+x+x^2+x^3=(1+x^2)(1+x)$. Let d be a common prime divisor of $1+x$ and $1+x^2$. Then d must also divide $x(x-1)=(1+x^2)-(1+x)$. Since $\gcd(x,x+1)=1$, d must divide $x-1$ and so divide $2=(x+1)-(x-1)$. Hence, the only common prime divisor of $1+x^2$ and $1+x$ is 2.

Suppose $1+x+x^2+x^3=(1+x^2)(1+x)$ is square. Then there are only two possibilities:

$$\text{(i)}\quad 1+x^2=u^2 \quad\text{and}\quad 1+x=v^2 \quad\text{for integers } u \text{ and } v\ ;$$

$$\text{(ii)}\quad 1+x^2=2r^2 \quad\text{and}\quad 1+x=2s^2 \quad\text{for integers } r \text{ and } s\ .$$

Ad (i): $1=u^2-x^2=(u-x)(u+x)\Leftrightarrow(x,u)=(0,\pm 1)$.
Ad (ii): We have $x^2-2r^2=-1$ which has solutions

$$(x,r)=(-1,1),(1,1),(7,5),(41,29),\cdots\ .$$

The complete set of solutions of $x^2-2r^2=\pm 1$ in positive integers is given by $\{(x_n,r_n):n=1,2,\cdots\}$, where $x_n+r_n\sqrt{2}=(1+\sqrt{2})^n$, with odd values of n yielding solutions of $x^2-2r^2=-1$. We need to select values of x for which $x+1=2s^2$ for some s. $x=-1,1,7$ work, yielding

$$1-1+(-1)^2+(-1)^3=0$$

$$1+1+1^2+1^3=2^2$$

$$1+7+7^2+7^3=8\times 50=20^2\ .$$

There may be other solutions.

7. (d) *First solution.* Let $f(x)=x^4+x^3+x^2+x+1=(x^5-1)/(x-1)$, with the quotient form for $x\neq 1$. We have that $f(0)=f(-1)=1^2$ and $f(3)=(243-1)/2=11^2$. Also $f(1)=5$ and $f(2)=31$. Suppose that $x\geq 4$. Then $x(x-2)>3$, so that $x^2>2x+3$. Hence

$$\begin{aligned}(2x^2+x+1)^2&=4x^4+4x^3+5x^2+2x+1\\&>4x^4+4x^3+4x^2+4x+4=4f(x)\end{aligned}$$

and

$$\begin{aligned}4f(x) &= (4x^4 + 4x^3 + x^2) + (3x^2 + 4x + 4)\\ &= (2x^2 + x)^2 + (3x^2 + 4x + 4) > (2x^2 + x)^2 \ .\end{aligned}$$

Thus, $4f(x)$ lies between the consecutive squares $(2x^2 + x)^2$ and $(2x^2 + x + 1)^2$ and so cannot be square. Hence $f(x)$ cannot be square.

Similarly, if $x \leq -2$, then $x(x-2) > 3$ and $3x^2 + 4x + 4 > 0$, and we again find that $4f(x)$ lies between the consecutive squares $(2x^2 + x)^2$ and $(2x^2 + x + 1)^2$. Hence $f(x)$ is square if and only if $x = -1, 0, 3$.

(d) *Second solution.* [M. Boase] For $x > 3$,

$$(x^2 + \frac{x}{2})^2 < x^4 + x^3 + x^2 + x + 1 < (x^2 + \frac{x+1}{2})^2$$

so that, lying between two half integers, $x^4 + x^3 + x^2 + x + 1$ is not square. Suppose $x = -y$ is less than -1. Since $y - 1 < \frac{3}{4}y^2$ and $y^2 + 2y - 3 = (y+3)(y-1) > 0$,

$$(y^2 - \frac{y}{2})^2 < 1 - y + y^2 - y^3 + y^4 < (y^2 - \frac{y-1}{2})^2$$

so again the middle term is not square. The cases $x = -1, 0, 1, 2, 3$ can be checked directly.

7. (e) *First solution.* Let

$$\begin{aligned}g(x) &= x^5 + x^4 + x^3 + x^2 + x + 1 = (x+1)(x^4 + x^2 + 1)\\ &= (x+1)[(x^2+1)^2 - x^2] = (x+1)(x^2 + x + 1)(x^2 - x + 1) \ .\end{aligned}$$

Observe that $g(x) < 0$ for $x \leq 2$, so $g(x)$ cannot be square in this case. Let us analyze common divisors of the three factors of $g(x)$.

Suppose that p is a prime divisor of $x + 1$. Then

$$x^2 + x + 1 = x(x+1) + 1 \equiv 1 \quad \bmod\ p$$

and

$$x^2 - x + 1 = x(x+1) - 2(x+1) + 3 \equiv 3 \quad \bmod\ p \ .$$

Hence $\gcd(x+1, x^2 + x + 1) = 1$ and $\gcd(x+1, x^2 - x + 1)$ is either 1 or 3.

Suppose q is prime and $x^2 + x + 1 \equiv 0 \pmod{q}$. Then $x(x+1) \equiv -1 \pmod{q}$, and $x^2 - x + 1 \equiv -2x \pmod{q}$. Since $x^2 + x + 1$ is odd, $q \neq 2$, then $x^2 - x + 1 \not\equiv 0 \pmod{q}$. Hence $\gcd(x^2 + x + 1, x^2 - x + 1) = 1$.

As we have seen from (b), $x^2 + x + 1$ is square if and only if $x = -1$ or 0. Indeed $g(-1) = 0^2$ and $g(0) = 1^2$. Otherwise, $x^2 + x + 1$ cannot be square. But $\gcd(x^2 + x + 1, (x+1)(x^2 - x + 1)) = 1$, so $g(x)$ cannot be a square either. Hence $x^5 + x^4 + x^3 + x^2 + x + 1$ is square if and only if $x = -1$ or 0.

(e) *Second solution.* [M. Boase] Observe that $x^5 + x^4 + \cdots + 1 = (x^3 + 1)(x^2 + x + 1)$. Since $x^3 + 1 = (x^2 + x + 1)(x - 1) + 2$, the greatest common divisor of $x^3 + 1$ and $x^2 + x + 1$ must divide 2. But $x^2 + x + 1 = x(x+1) + 1$ is always odd, so the greatest common divisor must be 1. Hence $x^2 + x + 1$ and $x + 1$ must both be square. Hence x must be either -1 or 0.

8.

8. *First solution.* Let $BF = \mu BA$, $BD = \lambda BC$ and $CE = \nu CA$.

The conditions are that

$$\lambda \leq \mu \leq \frac{1}{2} \qquad \text{and} \qquad 1 - \nu \leq 1 - \mu \quad \text{or} \quad \mu \leq \nu \ .$$

We observe that $[BDF] = \lambda\mu[ABC]$.

To see this, let $BG = \lambda BA$. Then

$$[BDF] = \frac{\mu}{\lambda}[BGD] = \frac{\mu}{\lambda}\lambda^2[ABC] = \mu\lambda[ABC] \ .$$

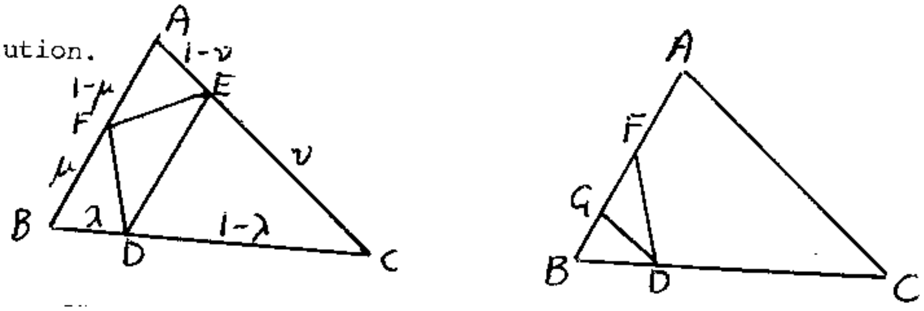
Similarly $[AFE] = (1-\mu)(1-\lambda)[ABC]$ and $[DEC] = \nu(1-\lambda)[ABC]$.

Hence

$$\begin{aligned}[DEF] &= (1 - \lambda\mu - (1-\mu)(1-\nu) - \nu(1-\lambda))[ABC] \\ &= (\mu - \mu\nu - \mu\lambda + \nu\lambda)[ABC] \\ &= \left(\frac{1}{4} - (\frac{1}{2} - \mu)^2 - (\mu - \lambda)(\nu - \mu)\right)[ABC] \leq \frac{1}{4}[ABC]\end{aligned}$$

with equality if and only if $\mu = 1/2$ and either $\lambda = \mu = 1/2$ or $\nu = \mu = 1/2$. The result follows.

First solution.



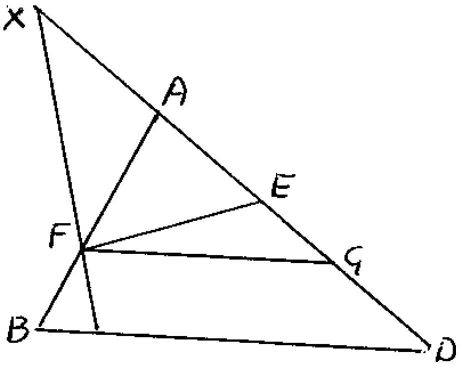
8. *Second solution.* Let G be on AC so that $FG||BC$. Then, since $\frac{AE}{EC} \leq \frac{AF}{FB}$, E lies in the segment AG.

Since $\frac{BD}{DC} \leq \frac{BF}{FA}$, DF produced is either parallel to AC or meets CA produced at a point X beyond A. Hence the distance from G to FD is not less than the distance from E to FD, so that $[DEF] \leq [FGD]$. The area of $[FGD]$ does not change as D varies along BC. To maximize $[DEF]$ is suffices to consider the special case of triangles $[FGD]$. Let $AF = xAB$. Then $FG = xBC$ and the heights of ΔDFG and ΔABC are in the ratio $1-x$. Hence

$$\frac{[DFG]}{[ABC]} = x(1-x)$$

which is maximized when $x = \frac{1}{2}$. The result follows from this, with $[DEF]$ being exactly one quarter of $[ABC]$ when F and G are the midpoints of AB and AC respectively.

Second solution



8. *Third solution.* Set up the situation as in the second solution. Let $BF = tFA$. Then $AB = (1+t)FA$, and the height of the triangle FGD is $t/(1+t)$ times the height of the triangle ABC. Hence

$$[DEF] \leq [FGD] = \frac{t}{(1+t)^2}[ABC] \ .$$

Now

$$\frac{1}{4} - \frac{t}{(1+t)^2} = \frac{(1-t)^2}{4(1+t)^2} \geq 0$$

so that $t(1+t)^{-2} \leq 1/4$ and the result follows. Equality occurs if and only if $t = 1$ and $E = G$, *i.e.*, F and E are both midpoints of their sides.

9.

9. *First solution.* Let Γ be the circle of equation $x^2 + y^2 = 1$ and let l be the line of equation $y = -1$. The points of the square must lie on the circle of equation $x^2 + y^2 = 2$. Let them be

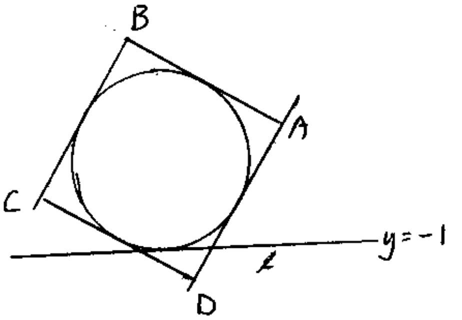
$$A \sim (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$$

$$B \sim (-\sqrt{2}\sin\theta, \sqrt{2}\cos\theta)$$
$$C \sim (-\sqrt{2}\cos\theta, -\sqrt{2}\sin\theta)$$
$$D \sim (\sqrt{2}\sin\theta, -\sqrt{2}\cos\theta)$$

for some angle θ with $-\pi/4 \leq \theta \leq \pi/4$. Observe that $1/\sqrt{2} \leq \cos\theta \leq 1$ and that $-1/\sqrt{2} \leq \sin\theta \leq 1/\sqrt{2}$. Then $A' \sim (\sqrt{2}\cos\theta, -1)$, $B' \sim (-\sqrt{2}\sin\theta, -1)$, $C' \sim (-\sqrt{2}\cos\theta, -1)$ and $D' \sim (\sqrt{2}\sin\theta, -1)$, so that $AA' = 1 + \sqrt{2}\sin\theta$, $BB' = 1 + \sqrt{2}\cos\theta$, $CC' = 1 - \sqrt{2}\sin\theta$ and $DD' = 1 - \sqrt{2}\cos\theta$. Hence

$$\begin{aligned} AA' \cdot CC' - BB' \cdot DD' &= (1 + \sqrt{2}\sin\theta)(1 - \sqrt{2}\sin\theta) - (1 + \sqrt{2}\cos\theta)|1 - \sqrt{2}\cos\theta| \\ &= (1 + \sqrt{2}\sin\theta)(1 - \sqrt{2}\sin\theta) + (1 + \sqrt{2}\cos\theta)(1 - \sqrt{2}\cos\theta) \\ &= 1 - 2\sin^2\theta + 1 - 2\cos^2\theta = 0 \quad . \end{aligned}$$

9. First solution



9. *Second solution.* One can proceed as in the first solution, taking the four points on the larger circle at the intersection with the perpendicular lines $y = mx$ and $y = -x/m$. The points are

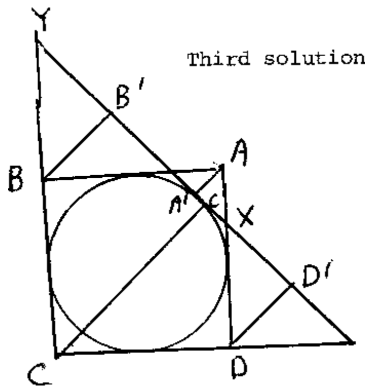
$$A \sim \left(\frac{\sqrt{2}}{\sqrt{m^2+1}}, \frac{m\sqrt{2}}{\sqrt{m^2+1}}\right) \qquad B \sim \left(\frac{-m\sqrt{2}}{\sqrt{m^2+1}}, \frac{\sqrt{2}}{\sqrt{m^2+1}}\right)$$

$$C \sim \left(\frac{\sqrt{2}}{\sqrt{m^2+1}}, \frac{-m\sqrt{2}}{\sqrt{m^2+1}}\right) \qquad D \sim \left(\frac{m\sqrt{2}}{\sqrt{m^2+1}}, \frac{-\sqrt{2}}{\sqrt{m^2+1}}\right) \quad .$$

In this case, the products turn out to be equal to $|(m^2 - 1)/(m^2 + 1)|$.

9. *Third solution.* [A. Birka] Let the circle have equation $x^2 + y^2 = 1$ and the square have vertices $A \sim (1, 1)$, $B \sim (-1, 1)$, $C \sim (-1, -1)$, $D \sim (1, -1)$. Suppose, wolog, that the line l is tangent to

the circle at $P(t, \sqrt{1-t^2})$ with $0 < t < 1$ and intersects CB produced in Y and AD in X. The line l has equation $tx + \sqrt{1-t^2}y = 1$ and so the coordinates of X are $(1, u)$ and of Y are $(-1, 1/u)$ where $u = (1-t)/\sqrt{1-t^2}$. Now $YB : YC = (1-u) : (1+u) = AX : XD$. Since $\Delta YBB'$ is similar to $\Delta YCC'$, $BB' : CC' = AA' : DD'$, and the result follows.

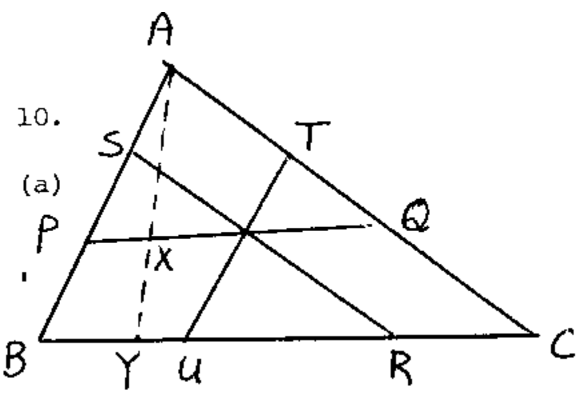


9. *Comment.* If the circle has equation $x^2 + y^2 = r^2$, the square has vertices $(\pm r, \pm r)$ and the line has equation through a point (a, b) on the circle has equation $ax + by = r^2$, then the distance product is $2ab$.

10.

10. (a) *First solution.* Recall that the centroid lies two-thirds of the way along the median from a vertex of the triangle to its opposite side. Let ABC be the fixed triangle and let $PQ \| BC$, $RS \| AC$ and $TU \| BA$, as in the diagram, with PQ, RS and TU intersecting in the centroid G.

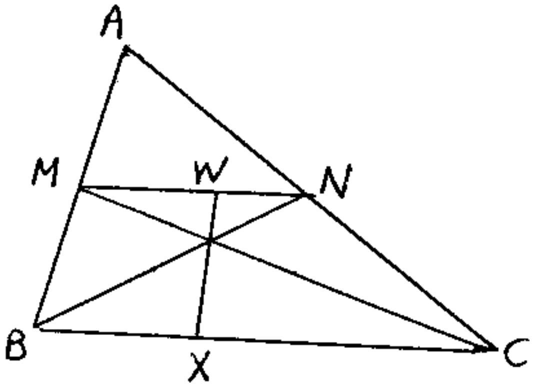
Observe, for example, that if X and Y lie on PQ and BC respectively, then $AX : XY = 2 : 3$. It follows from this that, if one ant is at A, then the centroid of the triangle formed by the three ants lies inside ΔAPQ (otherwise the midpoint of the side opposite the ant at A would not be in ΔABC). Similarly, if one ant is at B (respectively C) then the centroid of the ants' triangle lies within ΔBRS (respectively ΔCTU). Thus, if one ant traverses the entire perimeter, the centroid of the ants' triangle must lie inside the intersection of these three triangles, the singleton $\{G\}$. The result follows.



10. (b) *First solution.* Suppose the vertices of the triangle are given by the planar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$; the centroid of the triangle is at $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Suppose that one ant is placed at $t\mathbf{a} + (1-t)\mathbf{b}$ for $0 \le t \le 1$. Place the other two ants at $t\mathbf{b} + (1-t)\mathbf{c}$ and $t\mathbf{c} + (1-t)\mathbf{a}$. The centroid of the ants' triangle is at

$$\frac{1}{3}[(t\mathbf{a} + (1-t)\mathbf{b}) + (t\mathbf{b} + (1-t)\mathbf{c}) + (t\mathbf{c} + (1-t)\mathbf{a}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) \ .$$

(b)

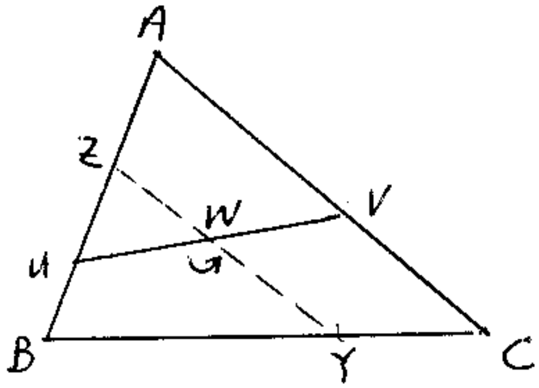


(b) *Second solution.* If one ant is at a vertex, then we can replace the remaining ants at the other vertices of the fixed triangle. Suppose, wolog, the ant is at X in the side BC.

Let MN be the line joining the midpoints M and N of AB and AC respectively; $MN \| BC$. Let XG meet MN at W. Since $BG : BN (= CG : CM) = 2 : 3$, it follows, by considering the similar triangles BGX and NGW, that $XG : XW = 2 : 3$. Hence the midpoint of the segment joining the other two ants' positions must be at W. Thus, the problem now is to find points Y and Z on the perimeter of ΔABC such that W is the midpoint of YZ. We use a continuity argument.

Let UV be any segment containing W whose endpoints lie on the perimeter of ΔABC. Let Y travel counterclockwise around the perimeter from U to V, and let Z be a point on the perimeter such that W lies on YZ. When Y is at U, $YW : WZ = VW : WV$, while when Y is at V, $YW : WZ = VW : WU$. Hence $YW : WZ$ varies continuously from a certain ratio to its reciprocal, so there must be a position for which $YW = WZ$.

(b)



(b) *Third solution.* [A. Panayotov] Suppose that the triangle has vertices at $(0,0)$, $(1,0)$ and (u,v), so that its centroid is at $(\frac{1}{3}(1+u), \frac{v}{3})$. Wolog, let one ant be at $(a,0)$ where $0 \leq a \leq 1$. Put the second ant at (u,v). Then we will place the third ant at a point $(b,0)$ on the $x-$axis. We require that $\frac{1}{3}(a+b+u) = \frac{1}{3}(1+u)$, so that $b = 1-a$. Clearly, $0 \leq b \leq 1$ and the result follows.

11.

11. *First solution.* We first establish that we must have $AD = BC$, $AB = CD$ and $AC = BD$. This is obvious if the triangular faces are equilateral. Suppose that ΔABC is isosceles with $AB = AC \neq BC$. If $AB = AD$, when $BC = BD$ and ΔBCD would fail to be congruent to ΔABC. Hence $AB = AC = BD = CD$. Suppose ΔABC is scalene. Since $\Delta ABC \equiv \Delta ABD$, AD must be equal to one of AB, AC or BC. Since ΔABD and ΔACD are not isosceles, the first two possibilities are excluded and so $AD = BC$. The other cases follow similarly. Fold out the tetrahedron flat to the configuation of the diagram.

Wolog, we can assign coordinates

$$A \sim (a,b,0) \qquad B \sim (-1,0,0) \qquad C \sim (1,0,0)$$

$$D_1 \sim (-a,-b,0) \qquad D_2 \sim (a+2,b,0) \qquad (a-2,b,0)$$

with a, b positive. Since the D_i swing up to D about the hinges BC, CA, AB, D must lie vertically above a point E in the $(x,y)-$plane for which $D_1E \perp BC$, $D_2E \perp AC$, $D_3E \perp AB$. This point E is given by

$$E \sim \left(-a, \frac{2a^2+b^2-2}{b}, 0 \right)$$

so that, for some value of t,

$$D \sim \left(-a, \frac{2a^2+b^2-2}{b}, t \right) \ .$$

Using the fact that $|AD| = 2$ and $|CD| = |CD_1|$, we find that

$$bt^2 = 4(1-a^2)(b^2+a^2-1) = 4(b^2+2a^2-1-a^2b^2-a^4) \ .$$

We check that the quantity on the right is indeed positive. Note that each angle of ΔABC is acute. For if $\angle A \geq 90°$, then $\angle D_3AB + \angle D_2AC \leq 90°$ and D_3A and D_2A could not fold up to coincide in a point above the plane. Since $\angle B < 90°$ and $\angle C < 90°$, A lies outside the circle diameter BC. Hence $a^2 < 1$ and $a^2 + b^2 > 1$.

Now $\overrightarrow{BC} = 2(1,0,0)$,

$$\overrightarrow{AD} = 2\left(-a, \frac{a^2-1}{b}, \frac{\sqrt{(1-a^2)(b^2+a^2-1)}}{b} \right) \ ,$$

and $|AD| = |BC| = 2$. Hence

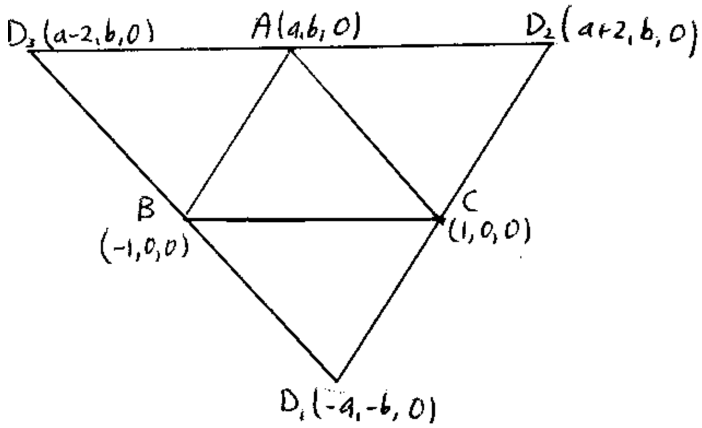
$$\cos\alpha = \frac{\overrightarrow{BC} \cdot \overrightarrow{AD}}{|BC||AD|} = -a \ .$$

Also $\sin(B+C) = \sin A$, so, by the Law of Sines,

$$\begin{aligned}
\frac{\sin(B-C)}{\sin(B+C)} &= \left(\frac{\sin B}{\sin A}\right)\cos C - \left(\frac{\sin C}{\sin A}\right)\cos B \\
&= \frac{|AC|}{2}\cos C - \frac{|AB|}{2}\cos B \\
&= \frac{(1-a)-(a+1)}{2} = -a = \cos\alpha \ .
\end{aligned}$$

[Note that $|AC|\cos C$ is the length of the projection of AC onto AB, namely $1-a$.]

11. First solution



11. *Second solution.* [A. Chan] Let $PQRS$ be the tetrahedron. $PQ = RS$, $PS = QR$, $PR = QS$. Let $B = \angle PQS = \angle PRS = \angle QPR = \angle QSR$ and $C = \angle SPQ = \angle SRQ = \angle PQR = \angle PSR$. Then

$$\begin{aligned}|PQ| \cdot |SR| \cos\alpha &= PQ \cdot SR = PQ \cdot (PS - PR) \\ &= PQ \cdot PS - PQ \cdot PR \\ &= |PQ||PS|\cos C - |PQ||PR|\cos B \quad .\end{aligned}$$

Therefore

$$|SR|\cos\alpha = |PS|\cos C - |PR|\cos B \quad .$$

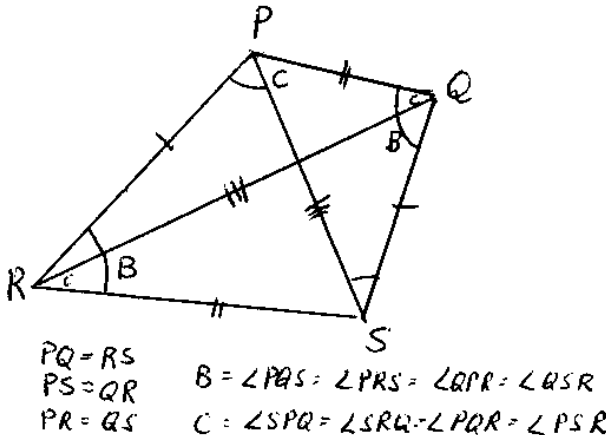
By the Law of Sines,

$$\frac{|PR|}{\sin C} = \frac{|PS|}{\sin B} = \frac{|SR|}{\sin(B+C)} \quad ,$$

whence

$$|SR|\cos\alpha = \frac{|SR|\sin B\cos C}{\sin(B+C)} - \frac{|SR|\cos B\sin C}{\sin(B+C)} = |SR|\frac{\sin(B-C)}{\sin(B+C)} \quad .$$

The result follows.



11. *Third solution.* [I. Panayotov] Assign coordinates $A \sim (u, v, 0)$, $B \sim (0, 0, 0)$, $C \sim (p, q, r)$ and $D \sim (w, 0, 0)$. Let $|AD| = |BC| = a$, $|BD| = |AC| = b$ and $|AB| = |CD| = c$. Suppose that α is the angle between AD and BC.

$$a^2\cos\alpha = (u - w, v, 0) \cdot (p, q, r) = pu - pw + qv \; .$$

Let $B = \angle ABC$ and $C = \angle CBD = \angle BCA$. Then

$$ac\cos B = (u, v, 0) \cdot (p, q, r) = pu + qv$$

and

$$ab\cos C = (w, 0, 0) \cdot (p, q, r) = pw \quad .$$

Then

$$a^2\cos\alpha - ac\cos B + ab\cos C = 0$$

whence

$$\begin{aligned}a\cos\alpha &= c\cos B - b\cos C \\ &= \frac{a\sin C\cos B - a\sin B\cos C}{\sin A} \\ &= \frac{a\sin(C-B)}{\sin(C+B)} \quad .\end{aligned}$$

12.

12. (a) *First solution.* $(a,b)=(3,2)$ yields $2a-1=5$, $2b-1=3$ and $a+b=5$; $(a,b)=(3,4)$ yields $2a-1=5$, $2b-1=7$ and $a+b=7$. Suppose that a and b are primes. Then for $a+b$ to be prime, $a+b$ must be odd, so that one of a and b, say b, is equal to 2. Thus, we require the $a+2$ and $2a-1$, along with a, to be prime. This is true when $a=3$.

Now suppose a is an odd prime exceeding 3. Then $a \equiv \pm 1 \pmod 6$, so the only way a and $a+2$ can both be prime is for $a \equiv -1 \pmod 6$, whence $2a-1 \equiv -3 \pmod 6$. Thus, 3 divides $2a-1$, and since $2a-1 \geq 9$, $2a-1$ must be composite.

12. (b) *First solution.* We first recall a bit of theory. Let p be a prime. By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod p$ whenever $\gcd(a,p)=1$. Let d be the smallest positive integer for which $a^d \equiv \pm 1 \pmod p$. Then d divides $p-1$, and indeed divides any positive integer k for which $a^k \equiv \pm 1 \pmod p$. Now to the problem.

Since $a+b$ is prime, $a \neq b$. Wolog, let $a > b$ and let $p = a+b$. Then $a \equiv -b \pmod p$, so that

$$a^b + b^a \equiv (-b)^b + b^a \equiv b^b((-1)^b + b^{a-b}) \ .$$

Suppose, if possible, that p divides a^b+b^a. Then, since $b < p$, $\gcd(b,p)=1$ and so $b^{a-b} \equiv (-1)^{b+1} \pmod p$. It follows that

$$b^{2b-1} = b^{(p-1)-(a-b)} \equiv (-1)^{b+1} \mod p \ .$$

Now $2b-1$ is prime, so that $2b-1$ must be the smallest exponent d for which $b^d \equiv \pm 1 \pmod p$. Hence $2b-1$ divides $a-b$, so that for some positive integer c, $a-b=c(2b-1)$, whence $a=b+2bc-c$ and so

$$2a-1 = 2b-1+(2b-1)2c = (2b-1)(2c+1) \ .$$

But $2a-1$ is prime and $2b-1>1$, so $2c+1=1$ and $c=0$. This is a contradiction. Hence p does not divide a^b+b^a.

Similarly, using the fact that $a^b+b^a \equiv (-b)^a + b^b \equiv b^b((-1)^a b^{a-b}+1)$, we can show that p does not divide a^a+b^b.

(b) *Second solution.* [M. Boase] Suppose that a and b exist as specified. Exactly one of a and b is odd, since $a+b$ is prime. Let it be a. Modulo $a+b$, we have that

$$0 \equiv a^b + b^a = a^b + (-a)^a \equiv a^b - a^a \equiv a^a(a^{b-a}-1) \ \text{ or } \ a^b(1-a^{a-b})$$

according as $a<b$ or $a>b$. Hence $a^{|b-a|}-1 \equiv 0 \pmod{a+b}$. Now $a+b-1 \pm |b-a| = 2a-1$ or $2b-1$, and $a^{a+b-1} \equiv 1 \pmod{a+b}$ (by Fermat's Little Theorem). Hence $a^{2a-1} \equiv a^{2b-1} \equiv 1 \pmod{a+b}$. Both $2a-1$ and $2b-1$ exceed 1 and are divisible by the smallest value of m for which $a^m \equiv 1 \pmod{a+b}$. Since both are prime, $2a-1=2b-1=m$, whence $a=b$, a contradiction. A similar argument can be applied to a^a+b^b.

(c) *Third solution.* Suppose, if possible, that one of a^b+b^a and a^a+b^b is divisible by $a+b$. Then $a+b$ divides their product $a^{a+b}+(ab)^a+(ab)^b+b^{a+b}$. By Fermat's Little Theorem, $a^{a+b}+b^{a+b} \equiv a+b \equiv 0 \pmod{a+b}$, so that $(ab)^a+(ab)^b \equiv 0 \pmod{a+b}$. Since $a+b$ is prime, it is odd and so $a \neq b$. Wolog, let $a>b$. Then

$$(ab)^a + (ab)^b = (ab)^b[(ab)^{a-b}+1]$$

and $\gcd(a,a+b)=\gcd(b,a+b)=1$, so that $(ab)^{a-b}+1 \equiv 0 \pmod{a+b}$. Since $(ab)^{a+b-1} \equiv 1 \pmod{a+b}$, it follows that $(ab)^{2a-1} \equiv (ab)^{2b-1} \equiv -1 \pmod{a+b}$. As in the foregoing solution, it follows that $a=b$, and we get a contradiction.