Mathematical Olympiads' Correspondence Program - 1997-1998

The following are the problems and solutions from the 1997-1998 Olympiads Correspondence Program operated for the Canadian Mathematical Society and the Mathematics Department of the University of Toronto by Ed Barbeau. Most (with luck, all) of the errors that appeared in previous versions have been corrected, but I would be grateful to receive further corrections. In writing up the solutions, I have attempted to follow up ideas submitted by the participants, and have acknowledged some of the authors. The diagrams appear after the solutions. In addition, problems from some of this year's Olympiads are appended as additional practice problems.

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Problems - Mathematical Olympiads' Correspondence Program - 1997-1998

Problem Set 1

1. (a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever the greatest common divisor of x and m is equal to 1. An example is $m = 12$. Suppose that n is a positive integer for which $n+1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

   (b) Does the result in (a) hold when $m = 2$?

   (c) Find all possible values of m that satisfy the condition in (a).

   (*Note:* (a) and (c) are posed as Problem 2229 in $CM + MM$, April, 1997. Part (c) was submitted by the proposer without a solution.

2. (a) Prove that, for each pair (m, n) of integers with $1 \leq m \leq n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \quad .$$

   (b) Suppose that $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, 3, \cdots, n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \leq i \leq r$, let t_i denote the ith smallest element in the subset, and let $T(n, r, i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n, r, i) = i\left(\frac{n+1}{r+1}\right) \quad .$$

   (*Note:* Part (b) is Problem 2239 in $CM + MM$, May, 1997.)

3. Let $0 < a < b$. Prove that, for any positive integer n,

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n + b^n}{2}} \quad .$$

4. How many distinct acute angles α are there for which $\cos\alpha\cos 2\alpha\cos 4\alpha = 1/8$?
   (*Note:* This is Problem 2249 in $CM + MM$, May, 1997.)

5. A convex closed figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

6. The nonisosceles right triangle ABC has $\angle CAB = 90°$. The inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle meets UV produced in S. Prove that

   (a) $ST \parallel BC$;

   (b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle and d_1 and d_2 are the respective distances from S to AC and AB.
   (*Note:* This is Problem 2235 in $CM + MM$.)

Solutions - Mathematical Olympiads' Correspondence Program - 1997-1998

1.

1. (a) *First solution.* Let $n+1$ be a multiple of m. Then $\gcd(m,n)=1$. We observe that n cannot be a square. Suppose, if possible, that $n=r^2$. Then $\gcd(r,m)=1$. Hence $r^2 \equiv 1 \pmod{m}$. But $r^2+1 \equiv 0 \pmod{m}$ by hypothesis, so that 2 is a multiple of m, a contradiction.

As a result, if d is a divisor of n, then n/d is a distinct divisor of n. Suppose $d|n$ (read "d divides n"). Since m divides $n+1$, therefore $\gcd(m,n)=\gcd(d,m)=1$, so that $d^2=1+bm$ for some integer b. Also $n+1=cm$ for some integer c. Hence

$$d+\frac{n}{d}=\frac{d^2+n}{d}=\frac{1+bm+cm-1}{d}=\frac{(b+c)m}{d} \quad .$$

Since $\gcd(d,m)=1$ and $d+n/d$ is an integer, d divides $b+c$ and so $d+n/d \equiv 0 \pmod{m}$.

Hence

$$\sum_{d|n} d = \sum \{(d+n/d) : d|n, d<\sqrt{n}\} \equiv 0 \pmod{m}$$

as desired.

(a) *Second solution.* Suppose that $n+1 \equiv 0 \pmod{m}$. As in the first solution, it can be established that n is not a perfect square. Let x be any positive divisor of n and suppose that $xy=n$; x and y are distinct. Since gcd $(x,m)=1$, $x^2 \equiv 1 \pmod{m}$, so that

$$y = x^2y \equiv xn \equiv -x \pmod{m}$$

whence $x+y$ is a multiple of m. Thus, the divisors of n comes in pairs, each of which has sum divisible by m, and the result follows.

(a) *Third solution.* [M. Boase] As in the second solution, if $xy=n$, then $x^2 \equiv y^2 \equiv 1 \pmod{m}$ so that

$$0 \equiv x^2-y^2 \equiv (x-y)(x+y) \pmod{m}.$$

For any divisor r of m, we have that

$$x(x-y) \equiv x^2-xy \equiv 1 \pmod{r}$$

from which it follows that the greatest common divisor of m and $x-y$ is 1. Therefore, m must divide $x+y$ and the solution can be completed as before.

1. (b) When $m=2$, the result does not hold. The hypothesis is true. However, the conclusion fails when $n=9$ since $9+1$ is a multiple of 2, but $1+3+9=13$ is odd.

1. (c) *First solution.* By inspection, we find that $m=1,2,3,4,6,8,12,24$ all satisfy the condition in (a). Suppose that m is odd. Then $\gcd(2,m)=1 \Rightarrow 2^2=4 \equiv 1 \pmod{m} \Rightarrow m=1,3$.

Suppose that m is not divisible by 3. Then $\gcd(3,m)=1 \Rightarrow 9=3^2 \equiv 1 \pmod{m} \Rightarrow m=1,2,4,8$. Hence any further values of m not listed in the above must be even multiples of 3, that is, multiples of 6.

Suppose that $m \geq 30$. Then, since $25=5^2 \not\equiv 1 \pmod{m}$, m must be a multiple of 5.

It remains to show that in fact m cannot be a multiple of 5. We observe that there are infinitely many primes congruent to 2 or 3 modulo 5. [To see this, let $q_1, \cdots, q_s$ be the s smallest odd primes of this form and let $Q=5q_1\cdots q_s+2$. Then Q is odd. Also, Q cannot be a product only of primes congruent to ± 1 modulo 5, for then Q itself would be congruent to ± 1. Hence Q has an odd prime factor congruent to ± 2 modulo 5, which must be distinct from q_1, $\cdots$, q_s. Hence, no matter how many primes we have of the desired form, we can always find one more.] If possible, let m be a multiple of 5 with the stated property and let q be a prime exceeding m congruent to ± 2 modulo 5. Then $\gcd(q,m)=1 \Rightarrow q^2 \equiv 1$

(mod m) $\Rightarrow q^2 \equiv 1 \pmod 5 \Rightarrow q \not\equiv \pm 2 \pmod 5$, yielding a contradiction. Thus, we have given a complete collection of suitable numbers m.

(c) *Second solution.* [D. Arthur] Suppose that $m = ab$ satisfies the condition of part (a), where the greatest common divisor of a and b is 1. Let gcd $(x, a) = 1$. Since a and b are coprime, there exists a number t such that $at \equiv 1 - x \pmod b$, so that $z = x + at$ and b are coprime. Hence, the greatest common divisor of z and ab equals 1, so that $z^2 \equiv 1 \pmod{ab}$, whence $x^2 \equiv z^2 \equiv 1 \pmod a$. Thus a (and also b) satisfies the condition of part (a).

When m is odd and exceeds 3, then gcd $(2, m) = 1$, but $2^2 = 4 \not\equiv 1 \pmod m$, so m does not satisfy the condition. When $m = 2^k$ for $k \geq 4$, then gcd $(3, m) = 1$, but $3^2 = 9 \not\equiv 1 \pmod m$. It follows from the first paragraph that if m satisfies the condition, it cannot be divisible by a power of 2 exceeding 8 nor by an odd number exceeding 3. This leaves the possibilities $1, 2, 3, 4, 6, 8, 12, 24$, all of which satisfy the condition.

2.

2. (a) *First solution.* $i(i-1)(i-2)\cdots(i-m+1) = \frac{[(i+1)-(i-m)]}{m+1} i(i-1)(i-2)\cdots(i-m+1)$

$$= \frac{(i+1)i(i-1)\cdots(i-m+1) - i(i-1)(i-2)\cdots(i-m+1)(i-m)}{m+1}$$

so that

$$\begin{aligned}\sum_{i=0}^{n} i(i-1)(i-2)\cdots(i-m+1) &= \sum_{i=2}^{n+1} \frac{i(i-1)\cdots(i-m)}{m+1} - \sum_{i=1}^{n} \frac{i(i-1)\cdots(i-m)}{m+1} \\ &= \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} - 0 \\ &= \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \quad .\end{aligned}$$

(a) *Second solution.* [K. Yeats] Let $n = m + k$. Then

$$\begin{aligned}\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) &= m! + \frac{(m+1)!}{1!} + \cdots + \frac{n!}{(n-m)!} \\ &= \frac{1}{(m+1)k!}\left[(m+1)!k! + \frac{(m+1)!k!(m+1)}{1!} + \frac{(m+2)!k!(m+1)}{2!} + \cdots + n!(m+1)\right] \\ &= \frac{(m+1)!}{(m+1)k!}\left[k! + \frac{k!}{1!}(m+1) + \frac{k!}{2!}(m+2)(m+1) + \cdots + n(n-1)\cdots(m+2)(m+1)\right] .\end{aligned}$$

The quantity in square brackets has the form (with $q = 0$)

$$\frac{k!}{q!} + \frac{k!}{(q+1)!}(m+1) + \frac{k!}{(q+2)!}(m+q+2)(m+1) + \frac{k!}{(q+3)!}(m+q+3)(m+q+2)(m+1) + \cdots$$

$$+\frac{k!}{k!}n(n-1)\cdots(m+q+2)(m+1)$$

$$= (m+q+2)\left[\frac{k!}{(q+1)!} + \frac{k!}{(q+2)!}(m+1) + \frac{k!}{(q+3)!}(m+q+3)(m+1) + \cdots + \frac{k!}{k!}n\cdots(m+q+3)(m+1)\right].$$

Applying this repeatedly with $q = 0, 1, 2, \cdots, k-1$ leads to the expression for the left sum in the problem of

$$\frac{(m+k+1)!}{(m+1)k!}\left[\frac{k!}{k!}\right] = \frac{(n+1)!}{(m+1)(n-m)!} = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \quad .$$

[A variant, due to D. Nicholson, uses an induction on r to prove that, for $m \leq r \leq n$,

$$\sum_{i=m}^{r} i(i-1)\cdots(i-m+1) = \frac{(r+1)!}{(r-m)!(m+1)} \ .]$$

(a) *Third solution.* For $1 \leq i \leq m-1$, $i(i-1)\cdots(i-m+1) = 0$. For $m \leq i \leq n$, $i(i-1)\cdots(i-m+1) = m!\binom{i}{m}$. Also,

$$\frac{(n+1)n\cdots(n-m+1)}{m+1} = m!\binom{n+1}{m+1}$$

so the statement is equivalent to

$$\sum_{m}^{n} \binom{i}{m} = \binom{n+1}{m+1} \ .$$

This is clear for $n = m$. Suppose it holds for $n = k \geq m$. Then

$$\sum_{i=m}^{k+1} \binom{i}{m} = \binom{k+1}{m+1} + \binom{k+1}{m} = \binom{k+2}{m+1}$$

and the result follows by induction.

(a) *Fourth solution.* Use induction on n. If $n = 1$, then $m = 1$ and both sides of the equation are equal to 1. Suppose that the result holds for $n = k$ and $1 \leq m \leq k$. Then, for $1 \leq m \leq k$,

$$\begin{aligned}\sum_{i=1}^{k+1} i(i-1)\cdots(i-m+1) &= \frac{(k+1)k(k-1)\cdots(k-m+1)}{m+1} + (k+1)k(k-1)\cdots(k-m+2) \\ &= \frac{(k+1)k(k-1)\cdots(k-m+2)}{m+1}[(k-m+1)+(m+1)] \\ &= \frac{(k+2)(k+1)k(k-1)\cdots(k-m+2)}{m+1}\end{aligned}$$

as desired. When $m = n = k+1$, all terms on the left have $k+1$ terms and so they vanish except for the one corresponding to $i = k+1$. This one is equal to $(k+1)!$ and so to the right side.

2. (b) *First solution.* For $1 \leq i \leq r \leq n$, let $S(n,r,i)$ be the sum of the elements t_i where $(t_1, t_2, \cdots, t_r)$ runs over r-tples with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$. Then $S(n,r,i) = \binom{n}{r}T(n,r,i)$. For $1 \leq k \leq n$, $1 \leq i \leq r$, the number of ordered $r-$tples $(t_1, t_2, \cdots, t_r)$ with $t_i = k$ is $\binom{k-1}{i-1}\binom{n-k}{r-i}$ where $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ when $b > a$. Hence

$$\binom{n}{r} = \sum_{k=1}^{n} \binom{k-1}{i-1}\binom{n-k}{r-i} \ .$$

Replacing n by $n+1$ and r by $r+1$ yields a reading

$$\binom{n+1}{r+1} = \sum_{k=1}^{n+1} \binom{k-1}{i-1}\binom{n+1-k}{r-(i-1)} \qquad \text{for} \ \ 1 \leq i \leq r+1 \ .$$

Replacing $i-1$ by i yields

$$\binom{n+1}{r+1} = \sum_{k=1}^{n+1} \binom{k-1}{i}\binom{n+1-k}{r-i} \qquad \text{for} \ \ 0 \leq i \leq r \ .$$

When $1 \le i \le r$, the first term of the sum is 0, so that

$$\binom{n+1}{r+1} = \sum_{k=2}^{n+1} \binom{k-1}{i} \binom{n-(k-1)}{r-i} = \sum_{k=1}^{n} \binom{k}{i} \binom{n-k}{r-i} \quad .$$

Thus

$$S(n,r,i) = \sum_{k=1}^{n} k \binom{k-1}{i-1} \binom{n-k}{r-i} = i \sum_{k=1}^{n} \binom{k}{i} \binom{n-k}{r-i} = i \binom{n+1}{r+1}$$

so

$$T(n,r,i) = i \frac{n+1}{r+1} \quad .$$

2. (b) *Second solution.* When $r = 1$, we have that

$$T(n,1,1) = \frac{1+2+\cdots+n}{n} = \frac{n+1}{2} \quad .$$

When $r = 2$, the subsets are $\{1,2\}, \{1,3\}, \cdots, \{1,n\}, \{2,3\}, \{2,4\}, \cdots \{2,n\}, \cdots, \{n-1,n\}$, so that

$$\begin{aligned} T(n,2,1) &= \frac{1 \times (n-1) + 2 \times (n-2) + \cdots + (n-1) \times 1}{\binom{n}{2}} \\ &= \frac{[(n-1)+(n-2)+\cdots+1] + [(n-2)+(n-3)+\cdots+1] + \cdots + 1}{\binom{n}{2}} \\ &= \frac{\sum_{j=1}^{n-1} [1+2+\cdots+(n-j)]}{n(n-1)/2} = \frac{\sum_{j=1}^{n-1} (n-j+1)(n-j)/2}{n(n-1)/2} \\ &= \frac{(1/6)(n+1)n(n-1)}{(1/2)n(n-1)} = \frac{n+1}{3} \quad , \end{aligned}$$

and

$$\begin{aligned} T(n,2,2) &= \frac{(n-1) \times n + (n-2) \times (n-1) + \cdots + 1 \times 2}{\binom{n}{2}} \\ &= \frac{(n+1)n(n-1)/3}{n(n-1)/2} = 2 \left(\frac{n+1}{3} \right) \quad . \end{aligned}$$

Thus, the result holds for $n = 1, 2$ and all i, r with $1 \le i \le r \le n$, and for all n and $1 \le i \le r \le 2$. Suppose as an induction hypothesis, we have established the result up to $n-1$ and all appropriate r and i, and for n and $1 \le i \le r-1$. The $r-$element subsets of $\{1, 2, \cdots, n\}$ have $\binom{n-1}{r}$ instances without n and $\binom{n-1}{r-1}$ instances with n.

Let $1 \le i \le r-1$. Then

$$\begin{aligned} T(n,r,i) &= \frac{\binom{n-1}{r} T(n-1,r,i) + \binom{n-1}{r-1} T(n-1,r-1,i)}{\binom{n}{r}} \\ &= \frac{i[\binom{n-1}{r} \frac{n}{r+1} + \binom{n-1}{r-1} \frac{n}{r}]}{\binom{n}{r}} = \frac{i[\binom{n}{r+1} + \binom{n}{r}]}{\binom{n}{r}} \\ &= i \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = i \frac{n+1}{r+1} \end{aligned}$$

Also

$$\begin{aligned} T(n,r,r) &= \frac{\binom{n-1}{r} T(n-1,r,r) + \binom{n-1}{r-1} n}{\binom{n}{r}} \\ &= \frac{\binom{n-1}{r} \frac{rn}{r+1} + \binom{n-1}{r-1} \frac{rn}{r}}{\binom{n}{r}} = \frac{r[\binom{n}{r+1} + \binom{n}{r}]}{\binom{n}{r}} = r \left(\frac{n+1}{r+1} \right) \quad . \end{aligned}$$

(b) *Third solution.* For $1 \leq i \leq r \leq n$, let $S(n,r,i)$ be the sum of the elements t_i where $(t_1, t_2, \cdots, t_r)$ runs over r-tples with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$. Then $S(n,r,i) = \binom{n}{r} T(n,r,i)$. We observe first that

$$S(n,r,i) = S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i)$$

for $1 \leq i \leq r-1$. This is true, since, for each j with $1 \leq j \leq n-r+1$, $S(n-j,r-1,i)$ adds the t_i over all $r-$tples for which $t_r = n-j+1$.

Now $S(n,1,1) = 1+2+\cdots+n = \frac{1}{2}(n+1)n$ and $S(n,2,1) = \frac{1}{2}n(n-1)+\cdots+1 = \frac{1}{3!}(n+1)n(n-1)$. As an induction hypothesis, suppose that $S(n,r-1,1) = \frac{1}{r!}(n+1)n(n-1)\cdots(n-r+2)$. Then

$$\begin{aligned} S(n,r,1) &= \sum_{k=r-1}^{n-1} S(k,r-1,1) \\ &= \frac{1}{r!}\sum_{k=r-1}^{n-1}(k+1)k(k-1)\cdots(k-r+2) = \frac{1}{r!}\sum_{k=1}^{n} k(k-1)\cdots(k-r+1) \\ &= \frac{1}{(r+1)!}(n+1)n(n-1)\cdots(n-r+1) = \binom{n+1}{r+1}\frac{n!}{r!(n-r)!} = \binom{n+1}{r+1}\binom{n}{r} . \end{aligned}$$

Thus, for each r with $1 \leq r \leq n$, $S(n,r,1) = \binom{n}{r}(n+1)/(r+1)$ so that $T(n,r,1) = (n+1)/(r+1)$.

Let $n \geq 2$. Suppose that for $1 \leq k \leq n-1$ and $1 \leq i \leq r \leq k$, it has been established that $S(k,r,i) = iS(k,r,1)$. Then for $1 \leq i \leq r \leq n$,

$$\begin{aligned} S(n,r,i) &= S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i) \\ &= i[S(n-1,r-1,1) + S(n-2,r-1,1) + \cdots + S(r-1,r-1,1) = iS(n,r,1) . \end{aligned}$$

Dividing by $\binom{n}{r}$ yields

$$T(n,r,i) = iT(n,r,1) = i\binom{n+1}{r+1} \quad .$$

Comments. (1) There is a one-one correspondence

$$(t_1, t_2, \cdots, t_r) \longleftrightarrow (n+1-t_r, n+1-t_{r-1}, \cdots, n+1-t_r)$$

of the set of suitable $r-$tples to itself, it follows that

$$\begin{aligned} S(n,r,r) &= \binom{n}{r}(n+1) - S(n,r,1) = \binom{n}{r}(n+1)\left[1 - \frac{1}{r+1}\right] \\ &= \frac{r(n+1)}{r+1}\binom{n}{r} = rS(n,r,1) \end{aligned}$$

from which $T(n,r,r) = r(n+1)/(r+1) = rT(n,r,1)$.

(2) To illustrate another method for getting and using the recursion, we prove first that $T(n,r,2) = 2T(n,r,1)$ for $2 \leq r \leq n$. Consider the case $r = 2$. For $1 \leq t_1 < t_2 \leq n$, $(t_1, t_2) \leftrightarrow (t_2 - t_1, t_2)$ defines a one-one correspondence between suitable pairs. Since $t_2 = t_1 + (t_2 - t_1)$, it follows from this correspondence that $S(n,2,2) = 2S(n,2,1)$. Dividing by $\binom{n}{2}$ yields $T(n,2,2) = 2T(n,2,1)$.

Suppose that $r \geq 2$. For each positive integer j with $1 \leq j \leq n-r+1$, we define a one-one correspondence between $r-$tples $(t_1, t_2, \cdots, t_r)$ with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$ and $t_3 - t_2 = j$ and $(r-1)-$tples $(s_1, s_2, s_3, \cdots, s_r) = (t_1, t_2, t_4 - j, \cdots, t_r - j)$ with $1 \leq s_1 = t_1 < s_2 = t_2 < s_3 = t_4 - j < \cdots < s_r =$

$t_r - j \leq n - j$. The sum of the elements t_2 over all $r-$tples with $t_3 - t_2 = j$ is equal to the sum of t_2 over all the $(r-1)-$tples. Hence

$$S(n,r,2) = S(n-1,r-1,2) + S(n-2,r-1,2) + \cdots + S(r-1,r-1,2) \quad .$$

More generally, for $1 \leq j \leq n-r-1$, there is a one-one correspondence between $r-$tples $(t_1, t_2, \cdots, t_r)$ with $t_{i+1} - t_i = j$ and $(r-1)-$tples $(s_1, s_2, \cdots, s_{r-1}) = (t_1, \cdots, t_i, t_{i+2} - j, \cdots, t_r - j)$ with $1 \leq s_1 = t_1 < \cdots < s_i = t_i < s_{i+1} = t_{i+2} - j < \cdots < s_{r-1} = t_r - j \leq n - j$. We now use induction on r. We have that

$$S(n,r,i) = S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i) \quad .$$

(b) *Fourth solution.* [Y. Shen] We establish that

$$\sum_{k=i}^{i+(n-r)} \binom{k}{i}\binom{n-k}{r-i} = \binom{n+1}{r+1} \quad .$$

Consider the $(r+1)-$element sets where $t_{i+1} = k+1$ and $t_{r+1} \leq n+1$. We must have $i \leq k \leq n-(r-i)$ and there are $\binom{k}{i}\binom{n-k}{r-i}$ ways of selecting $t_1, \cdots, t_i$ and $t_{i+2}, \cdots, t_{r+1}$. The desired equation follows from a counting argument over all possibilities for t_{i+1}.

In a similar way, we note that $t_i = k$ for $\binom{k-1}{i-1}\binom{n-k}{r-i}$ sets $\{t_1, \cdots, t_r\}$ chosen from $\{1, \cdots, n\}$, where $1 \leq k \leq n-r+1$. Observe that

$$\binom{k-1}{i-1}\binom{n-k}{r-i} = \frac{i}{k}\binom{k}{i}\binom{n-k}{r-i} \quad .$$

Then

$$\begin{aligned} T(n,r,i) &= \frac{\sum_{k=i}^{n-r+1} k\binom{k-1}{i-1}\binom{n-k}{r-i}}{\binom{n}{r}} \\ &= \frac{i\sum_{k=i}^{n-r+i}\binom{k}{i}\binom{n-k}{r-i}}{\binom{n}{r}} \\ &= \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = i\left(\frac{n+1}{r+1}\right) \quad . \end{aligned}$$

(b) *Fifth solution.* [Christopher So] Note that

$$\sum_{k=i}^{n-r+i} \binom{k}{i}\binom{n-k}{r-i}$$

is the coefficient of $x^i y^{r-i}$ in the polynomial

$$\sum_{k=i}^{n-r+i} (1+x)^k (1+y)^{n-k}$$

or in

$$\begin{aligned} \sum_{k=0}^{n+1} (1+x)^k (1+y)^{n-k} &= \frac{(1+y)^{n+1} - (1+x)^{n+1}}{y-x} \\ &= \frac{\sum_{j=0}^{n+1}\binom{n+1}{j}(y^j - x^j)}{y-x} \quad . \end{aligned}$$

Now the only summand which involves terms of degree r corresponds to $j = r+1$, so that the coefficient of $x^i y^{r-1}$ in the sum is the coefficient in the single term

$$\binom{n+1}{r+1}\frac{y^{r+1}-x^{r+1}}{y-x}$$

namely, $\binom{n+1}{r+1}$. We can now complete the argument as in the fourth solution.

(b) *Comment.* Let r and n be fixed values and consider i to be variable. The $\binom{n}{r}$ $r-$term sets contain altogether $r\binom{n}{r}$ numbers, each number occurring equally often: $\frac{r}{n}\binom{n}{r}$ times. The sum of all the elements in the set is

$$S(n,r,1)+S(n,r,2)+\cdots+S(n,r,r) = \frac{r}{n}\binom{n}{r}(1+2+\cdots+n) = \frac{r(n+1)}{2}\binom{n}{r}$$

where $S(n,r,i)$ is the sum of the elements t_i over the $\binom{n}{r}$ subsets. The ordered $r-$element subsets $(t_1,t_2,\cdots,t_r)$ can be mapped one-one to themselves by

$$(t_1,t_2,\cdots,t_r) \longleftrightarrow (n+1-t_r, n+1-t_{r-1},\cdots,n+1-t_1)\ .$$

From this, we see that, for $1 \le r$,

$$S(n,r,r+1-i) = \binom{n}{r}(n+1) - S(n,r,i)$$

so that

$$S(n,r,1)+S(n,r,r) = S(n,r,2)+S(n,r,r-1) = \cdots = S(n,r,i)+S(n,r,r+1-i) = \cdots = \binom{n}{r}(n+1)\ .$$

This is not enough to imply that $S(n,r,i)$ is an arithmetic progression in i, but along with this fact would give a quick solution to the problem.

3.

3. *First solution.* Dividing the inequality through by $(b+a)/2$ yields the equivalent inequality

$$1 \le \sqrt[n]{\frac{b'^{n+1}-a'^{n+1}}{(b'-a')(n+1)}} \le \sqrt[n]{\frac{a'^n+b'^n}{2}}$$

with $a' = (2a)/(b+a)$ and $b' = (2b)/(b+a)$. Note that $(a'+b')/2 = 1$, and we can write $b' = 1+u$ and $a' = 1-u$ with $0 < u < 1$. The central term becomes the nth root of

$$\begin{aligned}\frac{(1+u)^{n+1}-(1-u)^{n+1}}{2(n+1)u} &= \frac{2[(n+1)u+\binom{n+1}{3}u^3+\binom{n+1}{5}u^5+\cdots]}{2(n+1)u}\\ &= 1+\frac{1}{3}\binom{n}{2}u^2+\frac{1}{5}\binom{n}{4}u^4+\cdots\end{aligned}$$

which clearly exceeds 1 and gives the left inequality. The right term becomes the nth root of

$$\frac{1}{2}[(1+u)^n+(1-u)^n] = 1+\binom{n}{2}u^2+\binom{n}{4}u^4+\cdots$$

and the right inequality is clearly true.

3. *Second solution.* The inequality

$$\sqrt[n]{\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n+b^n}{2}}$$

is equivalent to

$$0 \leq (n+1)(a^n+b^n) - \frac{2(b^{n+1}-a^{n+1})}{b-a}.$$

The right side is equal to

$$\begin{aligned}
(n+1)(a^n+b^n) &- 2(b^n + b^{n-1}a + b^{n-2}a^2 \cdots + b^2a^{n-2} + ba^{n-1} + a^n) \\
&= (a^n - b^n) + (a^n - b^{n-1}a) + (a^n - b^{n-2}a) + \cdots + (a^n - ba^{n-1}) + (a^n - a^n) \\
&\qquad + (b^n - b^n) + (b^n - b^{n-1}a) + \cdots + (b^n - ba^{n-1}) + (b^n - a^n) \\
&= (a^n - b^n) + a(a^{n-1} - b^{n-1}) + a^2(a^{n-2} - b^{n-2}) + \cdots + a^{n-1}(a-b) + 0 \\
&\qquad + 0 + b^{n-1}(b-a) + \cdots + b(a^{n-1} - a^{n-1}) + (b^n - a^n) \\
&= 0 + (b-a)(b^{n-1} - a^{n-1}) + (b^2 - a^2)(b^{n-2} - a^{n-2}) + \cdots + (b^{n-1} - a^{n-1})(b-a) \\
&> 0 \ .
\end{aligned}$$

The left inequality

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}}$$

is equivalent to

$$\left(\frac{b+a}{2}\right)^n \leq \frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} \quad .$$

Let $v = \frac{1}{2}(b-a)$ so that $b + a = 2(a+v)$. Then

$$\begin{aligned}
\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} - \left(\frac{b+a}{2}\right)^n &= \frac{(a+2v)^{n+1}-a^{n+1}}{(b-a)(n+1)} - (a+v)^n \\
&= \frac{1}{2v(n+1)}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}a^{n+1-k}(2v)^k\right) - (a+v)^n \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}a^{n-(k-1)}(2v)^{k-1}\right) - (a+v)^n \\
&= \frac{1}{n+1}\left(\sum_{k=0}^{n}\binom{n+1}{k+1}a^{n-k}(2v)^k\right) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}v^k \\
&= \left(\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}a^{n-k}(2v)^k - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}v^k\right) \\
&= \sum_{k=0}^{n}\left(\frac{2^k}{k+1} - 1\right)\binom{n}{k}a^{n-k}v^k \geq 0 \quad ,
\end{aligned}$$

since $2^k = (1+1)^k = 1 + k + \binom{k}{2} + \cdots \geq 1 + k$ with equality if and only if $k = 0$ or 1. The result follows.

3. *Third solution.* [D. Nicholson] (partial) Let $n \geq 2i+1$. Then

$$(b^{n-i}a^i + b^ia^{n-i}) - (b^{n-i-1}a^{i+1} + b^{i+1}a^{n-i-1}) = (b-a)a^ib^i(b^{n-2i-1} - a^{n-2i-1}) \geq 0 \ .$$

Hence, for $0 \leq j \leq \frac{1}{2}(n+1)$,

$$b^n + a^n \geq b^{n-1}a + ab^{n-1} \geq \cdots \geq b^{n-j}a^j + b^j a^{n-j} \quad .$$

When $n = 2k+1$,

$$b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n = \sum_{i=0}^{k}(b^{n-i}a^i + b^i a^{n-i}) \leq (k+1)(b^n + a^n) = \frac{n+1}{2}(b^n + a^n)$$

and when $n = 2k$, we use the Arithmetic-Geometric Means Inequality to obtain $b^k a^k \leq \frac{1}{2}(a^{2k} + b^{2k})$, so that

$$b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n = \sum_{i=0}^{k-1}(b^{n-i}a^i + b^i a^{n-i}) + b^k a^k \leq k(b^n + a^n) + \frac{b^n + a^n}{2} = \frac{n+1}{2}(b^n + a^n) \ .$$

Hence

$$\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \leq \frac{b^n + a^n}{2} \ .$$

3. *Fourth solution.* [Y. Shen] Let $1 \leq k \leq n$ and $1 \leq i \leq k$. Then

$$(b^{k+1} + a^{k+1}) - (b^i a^{k+1-i} + a^i b^{k+1-i}) = (b^i - a^i)(b^{k+1-i} - a^{k+1-i}) \geq 0 \ .$$

Hence

$$k(b^{k+1} + a^{k+1}) \geq \sum_{i=1}^{k}(b^i a^{k+1-i} + a^i b^{k+1-i}) = 2\sum_{i=1}^{k} b^i a^{k+1-i} \quad .$$

This is equivalent to

$$\begin{aligned}
(2k+2)\sum_{i=0}^{k+1} b^i a^{k+1-i} &= (2k+2)(b^{k+1} + a^{k+1}) + (2k+2)\sum_{i=1}^{k} b^i a^{k+1-i} \\
&\geq (k+2)(b^{k+1} + a^{k+1}) + (2k+4)\sum_{i=1}^{k} b^i a^{k+1-i} \\
&= (k+2)(b^{k+1} + 2\sum_{i=1}^{k} b^i a^{k+1-i} + a^{k+1}) \\
&= (k+2)(b+a)\sum_{i=0}^{k} b^i a^{k-i}
\end{aligned}$$

which in turn is equivalent to

$$\frac{\sum_{i=0}^{k+1} b^i a^{k+1-i}}{k+2} \geq \frac{(b+a)(\sum_{i=0}^{k} b^i a^{k-i})}{2(k+1)} \quad .$$

We establish by induction that

$$\left(\frac{b+a}{2}\right)^n \leq \frac{1}{n+1}\sum_{i=0}^{n} b^i a^{n-i}$$

which will yield the left inequality. This holds for $n = 1$. Suppose that it holds for $n = k$. Then

$$\begin{aligned}
\left(\frac{b+a}{2}\right) &= \left(\frac{b+a}{2}\right) \cdot \left(\frac{b+a}{2}\right)^k \\
&\leq \left(\frac{b+a}{2}\right) \cdot \left(\frac{1}{k+1}\right)\sum_{i=0}^{k} b^i a^{k-i} \leq \frac{1}{k+2}\sum_{i=0}^{k+1} b^i a^{k-i} \quad .
\end{aligned}$$

As above, we have, for $k = n - 1$,

$$(n-1)(b^n + a^n) \geq 2 \sum_{i=1}^{n-1} b^i a^{n-i}$$

so that

$$(n+1)(b^n + a^n) \geq 2 \sum_{i=0}^{n} b^i a^{n-i} = 2\left(\frac{b^{n+1} - a^{n+1}}{b-a}\right)$$

from which the right inequality follows.

3. *Comment.* The inequality

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}}$$

is equivalent to

$$0 \leq \frac{2^n(b^{n+1} - a^{n+1})}{b-a} - (n+1)(b+a)^n \ .$$

When $n = 1$, the right side is equal to 0. When $n = 2$, it is equal to

$$4(b^2 + ba + a^2) - 3(b+a)^2 = (b-a)^2 > 0 \ .$$

When $n = 3$, we have

$$8(b^3 + b^2a + ba^2 + a^3) - 4(b+a)^3 = 4b^3 - 4b^2a - 4ba^2 + 4a^3 = 4(b^2 - a^2)(b-a) = 4(b+a)(b-a)^2 > 0 \ .$$

When $n = 4, 5$ and 6, the right side is, respectively,

$$(11b^2 + 18ab + 11a^2)(b-a)^2$$

$$(26b^3 + 54b^2a + 54ba^2 + 26a^3)(b-a)^2$$

$$(57b^4 + 136b^3a + 174b^2a^2 + 136ba^3 + 57a^4)(b-a)^2 \ .$$

There is a pattern here; can anyone express it in a general way that will yield the result, or at least show that the right side is the product of $(b-a)^2$ and a polynomial with positive coefficients?

4.

4. *First solution.*

$$8 \sin\alpha \cos\alpha \cos 2\alpha \cos 4\alpha = 4 \sin 2\alpha \cos 2\alpha \cos 4\alpha = 2 \sin 4\alpha \cos 4\alpha = \sin 8\alpha \ .$$

Hence the given equation is equivalent to $\sin 8\alpha = \sin\alpha$ with $\sin\alpha \neq 0$. Hence, we must have $\alpha + 8\alpha = \pi$, $8\alpha = 2\pi + \alpha$ or $8\alpha = 3\pi - \alpha$, since α lies strictly between 0 and $\pi/2$. These lead respectively to $\alpha = \pi/9$ ($20°$), $\alpha = 2\pi/7$ and $\alpha = \pi/3$ ($60°$). Thus, there are three acute angles α for which the equation is satisfied.

Comment. Alternatively, note that

$$0 = \sin 8\alpha - \sin\alpha = 2 \sin\frac{7\alpha}{2} \cos\frac{9\alpha}{2}$$

whence $7\alpha = 2\pi$ or $9\alpha = \pi$ or 3π.

Second solution. Let $x = \cos\alpha$. Then $\cos 2\alpha = 2x^2 - 1$ and $\cos 4\alpha = 2(2x^2 - 1)^2 - 1 = 8x^4 - 8x^2 + 1$, so that

$$\cos\alpha \cos 2\alpha \cos 4\alpha = x(2x^2 - 1)(8x^4 - 8x^2 + 1) \ .$$

Let $f(x) = x(2x^2 - 1)(8x^4 - 8x^2 + 1) - 1/8$. Using the fact that $\alpha = \pi/3$ corresponding to $x = 1/2$ is an obvious solution of the equation $f(x) = 0$, we find that

$$\begin{aligned} 8f(x) &= 128x^7 - 192x^5 + 80x^3 - 8x - 1 \\ &= (2x-1)(64x^6 + 32x^5 - 80x^4 - 40x^3 + 20x^2 + 10x + 1) \quad . \end{aligned}$$

Consider the factor

$$\begin{aligned} g(x) &= 64x^6 + 32x^5 - 80x^4 - 40x^3 + 20x^2 + 10x + 1 \\ &= (2x+1)(32x^5 - 40x^3 + 10x) + 1 \quad . \end{aligned}$$

By Descartes' Rule of Signs, $g(x) = 0$ has at most two positive roots (there are two changes of signs as one reads along the coefficients from left to right). Now $g(1) = 7$, $g(\frac{1}{2}) = 3$ and $g(\frac{3}{4}) = (\frac{5}{2})(\frac{-177}{32}) + 1 < 0$. Hence $g(x) = 0$ has precisely two roots, one in each of the intervals $(1/2, 3/4)$ and $(3/4, 1)$. These along with $x = 1/2$, constitute the three roots of $f(x) = 0$. Since all roots lie in $(0, 1)$, they are cosines of acute angles α for which $\cos\alpha\cos 2\alpha\cos 4\alpha = 1/8$.

Comment. $g(x) = (8x^3 + 4x^2 - 4x - 1)(8x^3 - 6x - 1)$. By Descartes' Rule of Signs, each factor has one positive root, and since each is negative for $x = 0$ and positive for $x = 1$, that root must lie between 0 and 1. We need to dispose of the possibility that both factors have the same root. If c is such a common root, then $8c^3 + 8c^2 - 4c - 1 = 8c^3 - 6c - 1 = 0 \Rightarrow 8c^2 + 2c = 0$ which yields a contradiction.

Third solution. [A. Birka] An obvious solution is $\alpha = \pi/3$. Observe that the product $\cos\alpha\cos 2\alpha\cos 4\alpha$ vanishes exactly when $\alpha = \pi/8$ and $\alpha = \pi/4$. On the interval $[0, \pi/8]$, $\cos\alpha$, $\cos 2\alpha$, $\cos 4\alpha$ are all nonnegative and decreasing, so the product decreases from 1 to 0. Hence, by the Intermediate Value Theorem and the continuity of the functions, $\cos\alpha\cos 2\alpha\cos 4\alpha = 1/8$ has precisely one solution in this interval. On the interval $(\pi/8, \pi/4)$, $\cos\alpha$ and $\cos 2\alpha$ are positive and $\cos 4\alpha$ is negative, so there are no solutions. Similarly, on the interval $(3\pi/8, \pi/2)$, $\cos\alpha$ and $\cos 4\alpha$ are positive and $\cos 2\alpha$ is negative, so there are no solutions. Hence any remaining solutions must lie in $(\pi/4, 3\pi/8)$, where $\cos\alpha > 0$, $\cos 2\alpha < 0$ and $\cos 4\alpha < 0$.

We need a preliminary result. Suppose that $u(\alpha)$ and $v(\alpha)$ are nonnegative twice differentiable functions on an open interval (a, b) such that $u(\alpha)$ is increasing, $v(\alpha)$ is decreasing and both functions are concave (*i.e.*, have negative second derivative). Then $u(\alpha)v(\alpha)$ is also concave. This follows immediately from $(uv)'' = u''v + 2u'v' + uv''$.

Apply this to the interval $(\frac{\pi}{4}, \frac{\pi}{3})$ and $u(\alpha) = -\cos 2\alpha$, $v(\alpha) = \cos\alpha$ to obtain that $u(\alpha)v(\alpha)$ is concave. Observe that $u(\alpha)v(\alpha)$ is nonnegative and

$$\begin{aligned} (uv)'(\alpha) &= [\cos\alpha(-\cos 2\alpha)]' = (\cos\alpha - 2\cos^3\alpha)' \\ &= \sin\alpha(6\cos^2\alpha - 1) \geq \frac{1}{2}\sin\alpha(6\cos^2\frac{\pi}{3} - 1) = \frac{1}{4}\sin\alpha > 0 \end{aligned}$$

so $u(\alpha)v(\alpha)$ is increasing.

On the interval $(\frac{\pi}{4}, \frac{\pi}{3})$, $-\cos 4\alpha$ is nonnegative, decreasing and concave, so we can apply the result again with $u(\alpha) = -\cos\alpha\cos 2\alpha$ and $v(\alpha) = -\cos 4\alpha$ to obtain that $f(\alpha) \equiv \cos\alpha\cos 2\alpha\cos 4\alpha$ is nonnegative and concave on $(\frac{\pi}{4}, \frac{\pi}{3})$. It can be checked that $f(\frac{\pi}{4}) = 0$, $f(\frac{\pi}{3}) = \frac{1}{8}$ and $f'(\frac{\pi}{3}) = -3\sqrt{3}/8 < 0$, so an analysis of the graph reveals exactly one solution of $f(\alpha) = 1/8$ for $\pi/4 < \alpha < \pi/3$.

Finally, we consider $\pi/3 < \alpha < 3\pi/8$. If $g(\alpha) \equiv \cos 2\alpha\cos 4\alpha = 2\cos^3 2\alpha - \cos 2\alpha$, then $g'(\alpha) = 2\sin 2\alpha[1 - 6\cos^2 2\alpha] < 0$, so $g(\alpha)$ is positive and decreasing on $(\frac{\pi}{3}, \frac{3\pi}{8})$. Since $g(\pi/3) = 1/4$, it follows that $f(\alpha) < (\cos\alpha)/4 < 1/8$ for $\pi/3 < \alpha < 3\pi/8$. The result follows.

Comment: It is interesting that the three solutions, given in increasing order of complexity, with the second requiring some sophisticated algebraic machinery and the last a reasonably sophisticated analysis, are also increasingly loath to provide information. The first solution is not only the easiest, but in fact allows us to explicitly state the values of α. In the second solution, we at least have the tools to make some kind of approximation to the solutions, although the one corresponding to the factor $(2x-1)$ can be given exactly. The analytic theory (intermediate value theorem) in the third solution just tells us that some interval has a solution without specifying very exactly where.

5.

5. *First solution.* Let the figure be denoted by $\mathcal{F}$ and the circle by $\mathcal{C}$, and let ρ be the central reflection through the centre of the circle. Suppose that m is any line of support for $\mathcal{F}$ and that it intersects the circle in P and Q. Then there are lines p and q through P and Q respectively, perpendicular to m, which support $\mathcal{F}$. Let p meet the circle in P and R, and q meet it in Q and S; let t be the line RS. Since $PQRS$ is concyclic with adjacent right angles, it is a rectangle, and t is a line of support of $\mathcal{F}$. Since PS and RQ are both diameters of $\mathcal{C}$, it follows that $S=\rho(P)$, $R=\rho(Q)$ and $t=\rho(m)$.

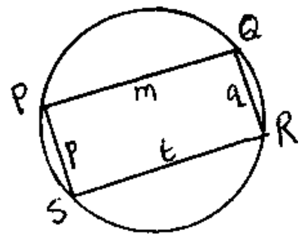
   Hence, every line of support of $\mathcal{F}$ is carried by ρ into a line of support of $\mathcal{F}$. We note that $\mathcal{F}$ must be on the same side of its line of support as the centre of the circle.

   Suppose that $X\in\mathcal{F}$. Let $Y=\rho(X)$. Suppose, if possible that $Y\notin\mathcal{F}$. Then there must be a disc containing Y that does not intersect $\mathcal{F}$, so we can find a line m of support for $\mathcal{F}$ such that $\mathcal{F}$ is on one side and Y is strictly on the other side of m. Let $n=\rho(m)$. Then n is a line of support for $\mathcal{F}$ which has $X=\rho(Y)$ on one side and $O=\rho(O)$ on the other. But this is not possible. Hence $Y\in\mathcal{F}$ and so $\rho(\mathcal{F})\subseteq\mathcal{F}$. Now $\rho\circ\rho$ is the identity mapping, so $\mathcal{F}=\rho(\rho(\mathcal{F}))\subseteq\rho(\mathcal{F})$. It follows that $\mathcal{F}=\rho(\mathcal{F})$ and the result follows.

   Second solution. Let P be any point on the circle $\mathcal{C}$. There are two perpendicular lines of support from P meeting the circle in Q and S. As in the first solution, we see that P is one vertex of a rectangle $PQRS$ each of whose sides supports $\mathcal{F}$. Let $\mathcal{G}$ be the intersection of all the rectangles as P ranges over the circumference of the circle $\mathcal{C}$. Since each rectangle has central symmetry about the centre of $\mathcal{C}$, the same is true of $\mathcal{G}$. It is clear that $\mathcal{F}\subseteq\mathcal{G}$. It remains to show that $\mathcal{G}\subseteq\mathcal{F}$. Suppose a point X in $\mathcal{G}$ does not belong to $\mathcal{F}$. Then there is a line r of support to $\mathcal{F}$ for which X and $\mathcal{F}$ are on opposite sides. This line of support intersects $\mathcal{C}$ at the endpoints of a chord which must be a side of a supporting rectangle for $\mathcal{F}$. The point X lies outside this rectangle, and so must lie outside of $\mathcal{G}$. The result follows.

   Third solution. [D. Arthur] If the result is false, then there is a line through the centre of the circle such that $OP>OQ$, where P is where the line meets the boundary of the figure on one side and Q is where it meets the boundary on the other. Let m be the line of support of the figure through Q. Then, as shown in Solution 1, its reflection t in the centre of the circle is also a line of support. But then P and O lie on opposite sides of t and we obtain a contradiction.

5.



6.

6. (a) *First solution.* Wolog, suppose that the situation is as diagrammed. $\angle BAC=\angle AUT=\angle AVT=90^\circ$, so that $AUVT$ is a rectangle with $AU=AV$ and $UT=VT$. Hence $AUTV$ is a square with diagonals AT and UV which right-bisect each other at W. Since SW right-bisects AT, by reflection in the line SW, we see that $\Delta ASU\equiv\Delta UST$, and so $\angle UTS=\angle UAS$.

   Let M be the midpoint of BC. Then M is the circumcentre of ΔABC, so that $MA=MC$ and

$\angle MCA = \angle MAC$. Since AS is tangent to the circumcircle of ΔABC, $AS \perp AM$. Hence

$$\angle UTS = \angle UAS = \angle SAM - \angle BAM = 90° - \angle BAM = \angle MAC = \angle MCA\ .$$

Now $UT \perp AB$ implies that $UT \| AC$. Since $\angle UTS = \angle ACB$, it follows that $ST \| BC$.

(a) *Second solution.* Wolog, suppose that S is on the opposite side of AB to C.

BT, being a part of the diameter produced of the inscribed circle, is a line of reflection that takes the circle to itself and takes the tangent BA to BC. Hence $\angle UBT = \frac{1}{2}\angle ABC$. Let $\alpha = \angle ABT$. By the tangent-chord theorem applied to the circumscribed circle, $\angle XAC = \angle ABC = 2\alpha$, so that $\angle SAU = 90° - 2\alpha$.

Consider triangles SAU and STU. Since $AUTV$ is a square (see the first solution), $AU = UT$ and $\angle AUV = \angle TUV = 45°$ so $\angle SUA = \angle SUT = 135°$. Also SU is common. Hence $\Delta SAU \equiv \Delta STU$, so $\angle STU = \angle SAU = 90° - 2\alpha$. Therefore,

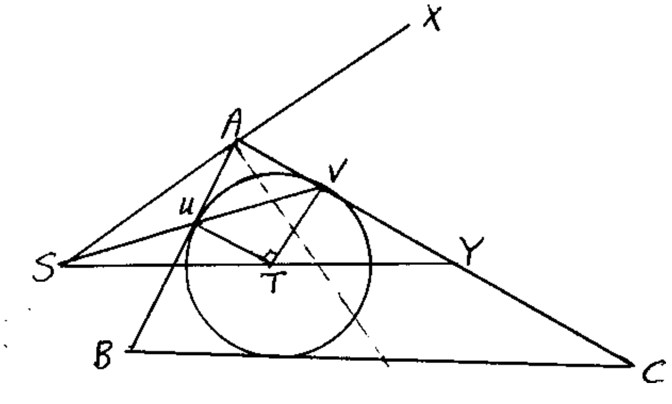
$$\angle STB = \angle UTB - \angle STU = (90° - \alpha) - (90° - 2\alpha) = \alpha = \angle TBC$$

from which it results that $ST \| BC$.

(a) *Third solution.* As before $\Delta AUS \equiv \Delta TUS$, so $\angle SAU = \angle STU$. Since $UT \| AC$, $\angle STU = \angle SYA$. Also, by the tangent-chord theorem, $\angle SAB = \angle ACB$. Hence $\angle SYA = \angle STU = \angle SAB = \angle ACB$, so $ST \| BC$.

(a) *Fourth solution.* In the Cartesian plane, let $A \sim (0,0)$, $B \sim (0,-b)$, $C \sim (c,0)$. The centre of the circumscribed circle is at $M \sim (c/2, -b/2)$. Since the slope of AM is $-b/c$, the equation of the tangent to the circumscribed circle through A is $y = (c/b)x$. Let r be the radius of the inscribed circle. Since $AU = AV$, the equation of the line UV is $y = x - r$. The abscissa of S is the solution of $x - r = (cx)/b$, so $S \sim (\frac{br}{b-c}, \frac{cr}{b-c})$. Since $T \sim (r, -r)$, the slope of ST is b/c and the result follows.

6. (a)



6. (b) *First solution.* $[\cdots]$ denotes area. Wolog, suppose that $d_1 > d_2$, as diagrammed.

Let r be the inradius of ΔABC. Then $[AVU] = \frac{1}{2}r^2$, $[AVS] = \frac{1}{2}rd_1$ and $[AUS] = \frac{1}{2}rd_2$. From $[AVU] = [AVS] - [AUS]$, it follows that $r^2 = rd_1 - rd_2$, whence $r = d_1 - d_2$.

(b) *Second solution.* [F. Crnogorac] Suppose that the situation is as diagrammed. Let P and Q be the respective feet of the perpendiculars from S to AC and AB. Since $\angle PVS = 45°$ and $\angle SPV = 90°$, ΔPSV is isosceles and so $PS = PV = PA + AV = SQ + AV$, *i.e.*, $d_1 = d_2 + r$.

(b) *Third solution.* Using the coordinates of the fourth solution of (a), we find that

$$d_1 = \left|\frac{cr}{b-c}\right| \qquad \text{and} \qquad d_2 = \left|\frac{br}{b-c}\right|$$

whence $|d_2 - d_1| = r$ as desired.

(b) *Fourth solution.* [M. Boase] Wolog, assume that the configuration is as diagrammed.

Since $\angle SUB = \angle AUV = 45°$, SU is parallel to the external bisector of $\angle A$. This bisector is the locus of points equidistant from AB and CA produced. Wolog, let PS meet this bisector in W, as in the diagram. Then $PW = PA$ so that $PS - PA = PS - PW = SW = AU$ and thus $d_1 - d_2 = r$.

6. (b)