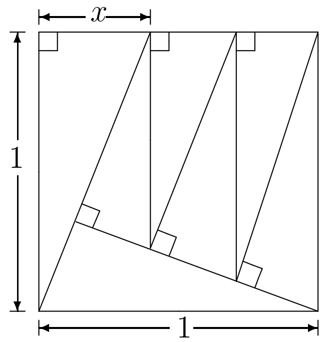
International Mathematical Talent Search – Round 8

Problem 1/8. Prove that there is no triangle whose altitudes are of length 4, 7, and 10 units.

Problem 2/8. As shown on the right, there is a real number x, $0 < x < 1$, so that the resulting configuration yields a dissection of the unit square into seven similar right triangles. This x must satisfy a monic polynomial of degree 5. Find that polynomial. (Note: A polynomial in x is monic if the coefficient of the highest power of x is 1.)



Problem 3/8. (i) Is it possible to rearrange the numbers 1, 2, 3, $\ldots$, 9 as $a(1)$, $a(2)$, $a(3)$, $\ldots$, $a(9)$ so that all the numbers listed below are different? Prove your assertion.

$$|a(1)-1|,\ |a(2)-2|,\ |a(3)-3|, \ldots,\ |a(9)-9|$$

(ii) Is it possible to rearrange the numbers 1, 2, 3, $\ldots$, 9, 10 as $a(1)$, $a(2)$, $a(3)$, $\ldots$, $a(9)$, $a(10)$ so that all the numbers listed below are different? Prove your assertion.

$$|a(1)-1|, |a(2)-2|, |a(3)-3|, \ldots, |a(9)-9|, |a(10)-10|$$

Problem 4/8. In a 50–meter run, Anita can give at most a 4–meter advantage to Bob and catch up with him by the finish line. In a 200–meter run, Bob can give at most a 15–meter advantage to Carol and catch up with her by the end of the race. Assuming that all three of them always proceed at a constant speed, at most how many meters of advantage can Anita give to Carol in a $1,000$–meter run and still catch up with her?

Problem 5/8. Given that a, b, x, and y are real numbers such that

$$\begin{aligned} a+b &= 23, \\ ax+by &= 79, \\ ax^2+by^2 &= 217, \\ ax^3+by^3 &= 691, \end{aligned}$$

determine ax^4+by^4.