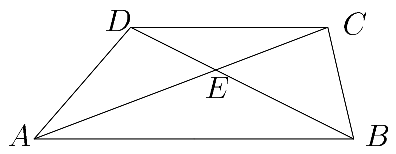
International Mathematical Talent Search – Round 7



Problem 1/7. In trapezoid $ABCD$, the diagonals intersect at E, the area of $\triangle ABE$ is 72, and the area of $\triangle CDE$ is 50. What is the area of trapezoid $ABCD$?

Problem 2/7. Prove that if a, b, and c are positive integers such that $c^2 = a^2 + b^2$, then both $c^2 + ab$ and $c^2 - ab$ are also expressible as the sums of squares of two positive integers.

Problem 3/7. For n a positive integer, denote by $P(n)$ the product of all positive integers divisors of n. Find the smallest n for which

$$P(P(P(n))) > 10^{12}.$$

Problem 4/7. In an attempt to copy down from the board a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers,

$$113, 137, 149, 155, 173,$$

accidentally omitting one. He later discovered that he also miscopied one of them. Can you help him and recover the original sequence?

Problem 5/7. Let $T = (a, b, c)$ be a triangle with sides a, b, and c and area $\triangle$. Denote by $T' = (a', b', c')$ the triangle whose sides are the altitudes of T (i.e., $a' = h_a$, $b' = h_b$, $c' = h_c$) and denote its area by $\triangle'$. Similarly, let $T'' = (a'', b'', c'')$ be the triangle formed from the altitudes of T', and denote its area by $\triangle''$. Given that $\triangle'$ = 30 and $\triangle'' = 20$, find $\triangle$.