International Mathematical Talent Search – Round 5

Problem 1/5. The set S consists of five integers. If pairs of distinct elements of S are added, the following ten sums are obtained: $1967, 1972, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991$. What are the elements of S?

Problem 2/5. Let $n \geq 3$ and $k \geq 2$ be integers, and form the forward differences of the members of the sequence

$$1, n, n^2, \ldots, n^{k-1}$$

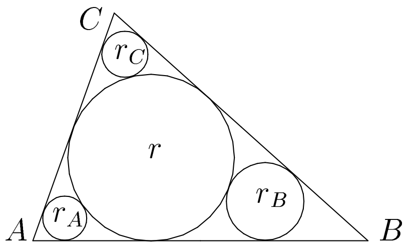
$$\begin{array}{ccccccccc}
1 & & 3 & & 9 & & 27 & & 81 \\
 & 2 & & 6 & & 18 & & 54 & \\
 & & 4 & & 12 & & 36 & & \\
 & & & 8 & & 24 & & & \\
 & & & & 16 & & & &
\end{array}$$

and successive forward differences thereof, as illustrated on the right for the case $(n, k) = (3, 5)$. Prove that all entries of the resulting triangle of positive integers are distinct from one another.

Problem 3/5. In a mathematical version of baseball, the umpire chooses a positive integer m, $m \leq n$, and you guess positive integers to obtain information about m. If your guess is smaller than the umpire's m, he calls it a "ball"; if it is greater than or equal to m, he calls it a "strike". To "hit" it you must state the correct value of m after the 3rd strike or the 6th guess, whichever comes first. What is the largest n so that there exists a strategy that will allow you to bat 1.000, i.e. always state m correctly? Describe your strategy in detail.

Problem 4/5. Prove that if f is a non-constant real-valued function such that for all real x, $f(x+1) + f(x-1) = \sqrt{3}f(x)$, then f is periodic. What is the smallest p. $p > 0$, such that $f(x+p) = f(x)$ for all x?

Problem 5/5. In $\triangle ABC$, shown on the right, let r denote the radius of the inscribed circle, and let r_A, r_B, and r_C denote the radii of the circles tangent to the inscribed circle and to the sides emanating from A, B, and C, respectively. Prove that



$$r \leq r_A + r_B + r_C.$$