International Mathematical Talent Search – Round 38

Problem 1/38. A well-known test for divisibility by 19 is as follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified on the right; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

$$\begin{array}{r}
6\,7\,9\,4\,\cancel{4} \\
8 \\
\hline
6\,8\,0\,\cancel{2} \\
4 \\
\hline
6\,8\,\cancel{4} \\
8 \\
\hline
7\,\cancel{6} \\
1\,2 \\
\hline
1\,9
\end{array}
\qquad\qquad
\begin{array}{r}
4\,4\,9\,7\,\cancel{6} \\
1\,2 \\
\hline
4\,5\,0\,\cancel{9} \\
1\,8 \\
\hline
4\,6\,\cancel{8} \\
1\,6 \\
\hline
6\,\cancel{2} \\
4 \\
\hline
1\,0
\end{array}$$

Find and prove a similar test for divisibility by 29.

Problem 2/38. Compute $1776^{1492!}$ (mod 2000); i.e., the remainder when $1776^{1492!}$ is divided by 2000. (As usual, the exclamation point denotes factorial.)

Problem 3/38. Given the arithmetic progression of integers

$$308, 973, 1638, 2303, 2968, 3633, 4298,$$

determine the unique geometric progression of integers,

$$b_1, b_2, b_3, b_4, b_5, b_6,$$

so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298.$$

Problem 4/38. Prove that every polyhedron has two vertices at which the same number of edges meet.

Problem 5/38. In $\triangle ABC$, segments PQ, RS, and TU are parallel to sides AB, BC, and CA, respectively, and intersect at the points X, Y, and Z, as shown in the figure on the right.

Determine the area of $\triangle ABC$ if each of the segments PQ, RS, and TU bisects (halves) the area of $\triangle ABC$, and if the area of $\triangle XYZ$ is one unit. Your answer should be in the form $a + b\sqrt{2}$, where a and b are positive integers.