International Mathematical Talent Search – Round 35

Problem 1/35. We define the *repetition number* of a positive integer n to be the number of distinct digits of n when written in base 10. Prove that each positive integer has a multiple which has a repetition number less than or equal to 2.

Problem 2/35. Let a be a positive real number, n a positive integer, and define the *power tower* $a \uparrow n$ recursively with $a \uparrow 1 = a$, $a \uparrow (i+1) = a^{a \uparrow i}$ for $i = 1, 2, \ldots$. For example, we have $4 \uparrow 3 = 4^{4^4} = 4^{256}$, a number which has 155 digits. For each positive integer k, let x_k denote the unique positive real number solution of the equation $x \uparrow k = 10 \uparrow (k+1)$. Which is larger: x_{42} or x_{43}?

Problem 3/35. Suppose that the 32 computers in a certain network are numbered with the 5-bit integers $00000, 00001, \ldots, 11111$ (bit is short for binary digit). Suppose that there is a one-way connection from computer A to computer B if and only if A and B share four of their bits with the remaining bit being a 0 at A and a 1 at B. (For example, 10101 can send messages to 11101 and to 10111.) We say that a computer is at level k in the network if it has exactly k 1's in its label ($k = 0, 1, 2, \ldots 5$). Suppose further that we know that 12 computers, three at each of the levels 1, 2, 3, and 4, are malfunctioning, but we do not know which ones. Can we still be sure that we can send a message from 00000 to 11111?

Problem 4/35. We say that a triangle in the coordinate plane is *integral* if its three vertices have integer coordinates and if its three sides have integer lengths.

(a) Find an integral triangle with a perimeter of 42.
(b) Is there an integral triangle with a perimeter of 43?

Problem 5/35. We say that a finite set of points is *well scattered* on the surface of a sphere if every open hemisphere (half the surface of the sphere without its boundary) contains at least one of the points. While $\{(1,0,0), (0,1,0), (0,0,1)\}$ is not well scattered on the unit sphere (the sphere of radius 1 centered at the origin), but if you add the correct point P, it becomes well scattered. Find, with proof, all possible points P that would make the set well scattered.