International Mathematical Talent Search – Round 31

Problem 1/31. Determine the three leftmost digits of the number

$$1^1 + 2^2 + 3^3 + \cdots + 999^{999} + 1000^{1000}.$$

Problem 2/31. There are infinitely many ordered pairs (m, n) of positive integers for which the sum

$$m + (m+1) + (m+2) + \cdots + (n-1) + n$$

is equal to the product mn. The four pairs with the smallest values of m are $(1, 1)$, $(3, 6)$, $(15, 35)$, and $(85, 204)$. Find three more (m, n) pairs.

Problem 3/31. The integers from 1 to 9 can be arranged into a 3×3 array so that the sum of the numbers in every row, column, and diagonal is a multiple of 9.

(a) Prove that the number in the center of the array must be a multiple of 3.

(b) Give an example of such an array with 6 in the center.

Problem 4/31. Prove that if $0 < x < \pi/2$, then

$$\sec^6 x + \csc^6 x + (\sec^6 x)(\csc^6 x) \geq 80.$$

Problem 5/31. In the figure shown on the right, O is the center of the circle, OK and OA are perpendicular to one another, M is the midpoint of OK, BN is parallel to OK, and $\angle AMN = \angle NMO$.

Determine the measure of $\angle ABN$ in degrees.