International Mathematical Talent Search – Round 30

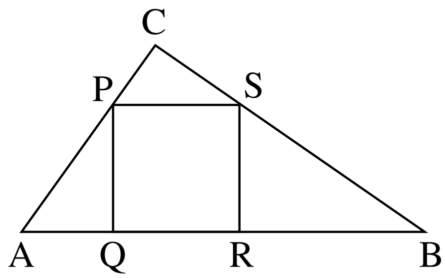
Problem 1/30. Determine the unique pair of real numbers (x, y) that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

Problem 2/30. Prove that there are infinitely many ordered triples of positive integers (a, b, c) such that the greatest common divisor of a, b, and c is 1, and the sum $a^2b^2 + b^2c^2 + c^2a^2$ is the square of an integer.

Problem 3/30. Nine cards can be numbered using positive half-integers $(1/2, 1, 3/2, 2, 5/2, \ldots)$ so that the sum of the numbers on a randomly chosen pair of cards gives an integer from 2 to 12 with the same frequency of occurence as rolling that sum on two standard dice. What are the numbers on the nine cards and how often does each number appear on the cards?

Problem 4/30. As shown in the figure on the right, square $PQRS$ is inscribed in right triangle ABC, whose right angle is at C, so that S and P are on sides BC and CA, respectively, while Q and R are on side AB. Prove that $AB \geq 3QR$ and determine when equality holds.



Problem 5/30. In the figure on the right, $ABCD$ is a convex quadrilateral, K, L, M, and N are the midpoints of its sides, and $PQRS$ is the quadrilateral formed by the intersections of AK, BL, CM, and DN. Determine the area of quadrilateral $PQRS$ if the area of quadrilateral $ABCD$ is 3000, and the areas of quadrilaterals $AMQP$ and $CKSR$ are 513 and 388, respectively.