International Mathematical Talent Search – Round 27

Problem 1/27. Are there integers M, N, K, such that $M + N = K$ and

(i) each of them contains each of the seven digits $1, 2, 3, \ldots, 7$ exactly once?

(ii) each of them contains each of the nine digits $1, 2, 3, \ldots, 9$ exactly once?

Problem 2/27. Suppose that $R(n)$ counts the number of representations of the positive integer n as the sum of the squares of four non-negative integers, where we consider two representations equivalent if they differ only in the order of the summands. (For example, $R(7) = 1$ since $2^2 + 1^2 + 1^2 + 1^2$ is the only representation of 7 up to ordering.)

Prove that if k is a positive integer, then $R(2^k) + R(2^{k+1}) = 3$.

Problem 3/27. Assume that $f(1) = 0$, and that for all integers m and n,

$$f(m + n) = f(m) + f(n) + 3(4mn - 1).$$

Determine $f(19)$.

Problem 4/27. In the rectangular coordinate plane, $ABCD$ is a square, and $(31, 27)$, $(42, 43)$, $(60, 27)$, and $(46, 16)$ are points on its sides, AB, BC, CD, and DA, respectively. Determine the area of $ABCD$.

Problem 5/27. Is it possible to construct in the plane the midpoint of a given segment using compasses alone (i.e., without using a straight edge, except for drawing the segment)?