International Mathematical Talent Search – Round 26

Problem 1/26. Assume that x, y, and z are positive real numbers that satisfy the equations given on the right.

$$\begin{aligned} x+y+xy &= 8, \\ y+z+yz &= 15, \\ z+x+zx &= 35. \end{aligned}$$

Determine the value of $x+y+z+xyz$.

Problem 2/26. Determine the number of non-similar regular polygons each of whose interior angles measures an integral number of degrees.

Problem 3/26. Substitute different digits $(0, 1, 2, \ldots, 9)$ for different letters in the alphametics on the right, so that the corresponding addition is correct, and the resulting value of M O N E Y is as large as possible. What is this value?

$$\begin{array}{cccccc} & & \mathrm{S} & \mathrm{H} & \mathrm{O} & \mathrm{W} \\ & & & & \mathrm{M} & \mathrm{E} \\ + & & & \mathrm{T} & \mathrm{H} & \mathrm{E} \\ \hline & \mathrm{M} & \mathrm{O} & \mathrm{N} & \mathrm{E} & \mathrm{Y} \end{array}$$

Problem 4/26. Prove that if $a \geq b \geq c > 0$, then

$$2a+3b+5c-\frac{8}{3}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \leq \frac{1}{3}\left(\frac{a^2}{b}+\frac{b^2}{c}+4\frac{c^2}{a}\right).$$

Problem 5/26. Let $ABCD$ be a convex quadrilateral inscribed in a circle, let M be the intersection point of the diagonals of $ABCD$, and let E, F, G, and H be the feet of the perpendiculars from M to the sides of $ABCD$, as shown in the figure on the right. Determine (with proof) the center of the circle inscribable in quadrilateral $EFGH$.