International Mathematical Talent Search – Round 24

Problem 1/24. The lattice points of the first quadrant are numbered as shown in the diagram on the right. Thus, for example, the 19th lattice point is $(2, 3)$, while the 97th lattice point is $(8, 5)$. Determine, with proof, the 1997th lattice point in this scheme.

○	○	○	○	○	○	○	○
22	35	○	○	○	○	○	○
21	23	34	○	○	○	○	○
11	20	24	33	○	○	○	○
10	12	19	25	32	○	○	○
4	9	13	18	26	31	○	○
3	5	8	14	17	27	30	○
1	2	6	7	15	16	28	29

Problem 2/24. Let $N_k = 131313\ldots131$ be the $(2k+1)$–digit number (in base 10), formed from $k+1$ copies of 1 and k copies of 3. Prove that N_k is not divisible by 31 for any value of $k = 1, 2, 3, \ldots$.

Problem 3/24. In $\triangle ABC$, let $AB = 52$, $BC = 64$, $CA = 70$, and assume that P and Q are points chosen on sides AB and AC, respectively, so that $\triangle APQ$ and quadrilateral $PBCQ$ have the same area and the same perimeter. Determine the square of the length of the segment PQ.

Problem 4/24. Determine the positive integers $x < y < z$ for which

$$\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{97}.$$

Problem 5/24. Let P be a convex planar polygon with n vertices, and from each vertex of P construct perpendiculars to the $n-2$ sides (or extensions thereof) of P not meeting at that vertex. Prove that either one of these perpendiculars is completely in the interior of P or it is a side of P.