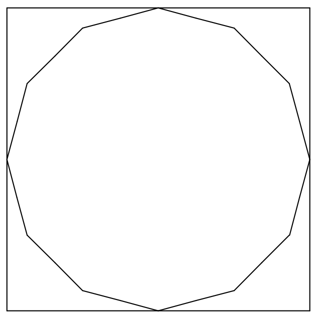
International Mathematical Talent Search – Round 12

Problem 1/12. A teacher writes a positive integer less than fifty thousand on the board. One student claims it is an exact multiple of 2; a second student says it is an exact multiple of 3; and so on, until the twelfth student says that it is an exact multiple of 13. The teacher observes that all but two of the students were right, and that the two students making incorrect statements spoke one after the other. What was the number written on the board?

Problem 2/12. A regular dodecagon is inscribed in a square of area 24 as shown on the right, where four vertices of the dodecagon are at the midpoints of the sides of the square. Find the area of the dodecagon.



Problem 3/12. Let S be a set of 30 points in the plane, with the property that the distance between any pair of distinct points in S is at least 1. Define T to be a largest possible subset of S such that the distance between any pair of distinct points in T is at least $\sqrt{3}$. How many points must be in T?

Problem 4/12. Prove that if $\sqrt[3]{2} + \sqrt[3]{4}$ is a zero of a cubic polynomial with integer coefficients, then it is the only real zero of that polynomial.

Problem 5/12. In the figure on the right, ℓ_1 and ℓ_2 are parallel lines, AB is perpendicular to them, and P, Q, R, S are the intersection points of ℓ_1 and ℓ_2 with a circle of diameter greater than $\overline{AB}$ and center, C, on the segment AB. Prove that the product $\overline{PR} \cdot \overline{PS}$ is independent of the choice of C on the segment AB.