International Mathematical Talent Search – Round 10

Problem 1/10. Find $x^2 + y^2 + z^2$ if x, y, and z are positive integers such that

$$7x^2 - 3y^2 + 4z^2 = 8 \text{ and } 16x^2 - 7y^2 + 9z^2 = -3.$$

Problem 2/10. Deduce from the simple estimate, $1 < \sqrt{3} < 2$, that $6 < 3^{\sqrt{3}} < 7$.

Problem 3/10. For each positive integer n, $n \geq 2$, determine a function

$$f_n(x) = a_n + b_n x + c_n |x - d_n|,$$

where a_n, b_n, c_n, d_n depend only on n, such that

$$f_n(k) = k + 1 \text{ for } k = 1, 2, \ldots, n-1 \text{ and } f_n(n) = 1.$$

Problem 4/10. A bag contains 1993 red balls and 1993 black balls. We remove two balls at a time repeatedly and

(i) discard them if they are of the same color,
(ii) discard the black ball and return to the bag the red ball if they are different colors.

What is the probability that this process will terminate with one red ball in the bag?

Problem 5/10. Let P be a point on the circumcircle of $\triangle ABC$, distinct from A, B, and C. Suppose BP meets AC at X, and CP meets AB at Y. Let Q be the point of intersection of the circumcircles of $\triangle ABC$ and $\triangle AXY$, with $Q \neq A$. Prove that PQ bisects the segment XY. (The various points of intersection may occur on the extensions of the segments.)